

# 目 录

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1.  $\sum_{n=1}^{\infty} \left[ \frac{1}{k(n-1)+a} - \frac{1}{k(n-1)+b} \right] \dots\dots\dots ( 25 )$

2.  $\sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{B-A}{k(n-1)+d} \right] \dots\dots\dots ( 68 )$

3.  $\sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{D}{k(n-1)+d} - \frac{A+D-B}{k(n-1)+e} \right] \dots\dots\dots ( 95 )$

4.  $\sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{D}{k(n-1)+d} - \frac{E}{k(n-1)+e} + \frac{B+E-A-D}{k(n-1)+f} \right] \dots\dots\dots (118)$

5.  $\sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{D}{k(n-1)+d} - \frac{E}{k(n-1)+e} + \frac{F}{k(n-1)+f} - \frac{A+D+F-B-E}{k(n-1)+g} \right] \dots\dots\dots (143)$

$$6. \sum_{n=1}^{\infty} \left[ \frac{A}{k(A^n - A) + a} - \frac{B}{k(B^n - B) + b} \right] \dots\dots (165)$$

$$7. \sum_{n=1}^{\infty} \left[ \frac{A}{k(A_1 n - A_1) + a} - \frac{B}{k(B_1 n - B_1) + b} \right. \\ \left. + \frac{D}{k(D_1 n - D_1) + d} \right] \dots\dots\dots (221)$$

$$8. \sum_{n=1}^{\infty} \left[ \frac{A}{k(A_1 n - A_1) + a} - \frac{B}{k(B_1 n - B_1) + b} \right. \\ \left. + \frac{D}{k(D_1 n - D_1) + d} - \frac{E}{k(E_1 n - E_1) + e} \right] \dots (257)$$

#### 四、一般的结果

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(mn - m) + a} + \frac{1}{k(mn - m + 1) + a} \right. \\ \left. + \frac{1}{k(mn - m + 2) + a} + \dots + \frac{1}{k(mn - 2) + a} \right. \\ \left. - \frac{m-1}{k(mn - 1) + a} \right] \dots\dots\dots (287)$$

## 一、常数 $C(a, k)$ 的定义及其性质

**定理1** 设  $a$  与  $k$  都是正整数, 则

$$\begin{aligned} \frac{1}{k} \left[ \ln(nk + a) - \ln a \right] &< \sum_{i=1}^n \frac{1}{k(i-1) + a} \\ &\leq \frac{1}{a} + \frac{1}{k} \left[ \ln(nk - k + a) - \ln a \right]. \end{aligned}$$

**证明** 数列  $\left\{ \left(1 + \frac{1}{n}\right)^n \right\}$  单调递增而趋于  $e$ , 所以

$$\left(1 + \frac{1}{n}\right)^n < e, \quad n \ln \frac{n+1}{n} < 1,$$

$$\ln(n+1) - \ln n < \frac{1}{n}. \quad (n=1, 2, \dots)$$

令  $n=i, i+1, \dots, i+k-1$ , 然后各式相加得:

$$\ln(i+k) - \ln i < \frac{1}{i} + \frac{1}{i+1} + \dots + \frac{1}{i+k-1} \leq \frac{k}{i},$$

$$\frac{1}{k} \left[ \ln(i+k) - \ln i \right] < \frac{1}{i}. \quad (i=1, 2, \dots)$$

令  $i=a, k+a, 2k+a, \dots, k(n-1)+a$ , 然后各式相加得:

$$\frac{1}{k} \left[ \ln(nk + a) - \ln a \right] < \sum_{i=1}^n \frac{1}{k(i-1) + a}.$$

数列  $\left\{ \left(1 + \frac{1}{n}\right)^{n+1} \right\}$  单调递减而趋于  $e$ , 所以

$$e < \left(1 + \frac{1}{n}\right)^{n+1}, \quad 1 < (n+1) \ln \frac{n+1}{n},$$

$$\frac{1}{n+1} < \ln(n+1) - \ln n. \quad (n=1, 2, \dots)$$

令  $n=i, i+1, \dots, i+k-1$ , 然后各式相加得

$$\frac{1}{i+1} + \frac{1}{i+2} + \dots + \frac{1}{i+k} < \ln(i+k) - \ln i,$$

$$\frac{1}{i+k} < \frac{1}{k} [\ln(i+k) - \ln i]. \quad (i=1, 2, \dots) \quad (*)$$

令  $i=a, k+a, 2k+a, \dots, k(n-2)+a$ , 然后各式相加得

$$\sum_{i=1}^n \frac{1}{k(i-1)+a} < \frac{1}{k} [\ln(nk-k+a) - \ln a].$$

$$\text{于是 } \sum_{i=1}^n \frac{1}{k(i-1)+a} \leq \frac{1}{a} + \frac{1}{k} [\ln(nk-k+a) - \ln a].$$

**定理2** 设  $a$  与  $k$  都是正整数, 则

$$\sum_{i=1}^n \frac{1}{k(i-1)+a} = C(a, k) + \frac{1}{k} \ln \frac{k(n-1)+a}{a} + \varepsilon_n(a, k),$$

其中  $C(a, k)$  是常数, 当  $n \rightarrow \infty$  时  $\varepsilon_n(a, k) \rightarrow 0$ .

为了方便, 记  $H_n(a, k) = \sum_{i=1}^n \frac{1}{k(i-1)+a}$ , 则

$$H_n(a, k) = C(a, k) + \frac{1}{k} \ln \frac{k(n-1)+a}{a} + \varepsilon_n(a, k).$$

**证明** 于(\*)令  $i=nk-k+a$ , 则有

$$\frac{1}{nk+a} < \frac{1}{k} \ln \frac{nk+a}{nk-k+a}. \quad (n=1, 2, \dots)$$

$$\text{令 } X_n = H_n(a, k) - \frac{1}{k} \ln \frac{k(n-1)+a}{a},$$

$$\text{则 } X_{n+1} = H_{n+1}(a, k) - \frac{1}{k} \ln \frac{nk+a}{a}.$$

$$\text{于是 } X_{n+1} - X_n = \frac{1}{nk+a} - \frac{1}{k} \ln \frac{nk+a}{nk-k+a} < 0.$$

$$(n=1, 2, \dots)$$

根据定理 1, 有

$$\begin{aligned} X_n &= H_n(a, k) - \frac{1}{k} \ln \frac{k(n-1)+a}{a} \\ &> \frac{1}{k} \ln \frac{nk+a}{nk-k+a} > 0. \quad (n=1, 2, \dots) \end{aligned}$$

已证明数列  $\{X_n\}$  单调递减而且下方有界, 所以极限  $\lim_{n \rightarrow \infty} X_n$

存在. 设  $\lim_{n \rightarrow \infty} X_n = C(a, k)$ , 即

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left[ H_n(a, k) - \frac{1}{k} \ln \frac{k(n-1)+a}{a} \right] \\ &= C(a, k), \end{aligned} \quad (**)$$

$$\text{则 } H_n(a, k) = C(a, k) + \frac{1}{k} \ln \frac{k(n-1)+a}{a} + \varepsilon_n(a, k),$$

当  $n \rightarrow \infty$  时  $\varepsilon_n(a, k) \rightarrow 0$ .

上述极限  $(**)$  就是常数  $C(a, k)$  的定义.

常数  $C(a, k)$  具有如下的两个性质:

**命题 1** 设  $m$  为正整数, 则

$$C(ma, mk) = \frac{1}{m} C(a, k).$$

$$\text{证明 } H_n(ma, mk) = \frac{1}{m} H_n(a, k).$$

根据定理 2, 有

$$C(ma, mk) + \frac{1}{mk} \ln \frac{mk(n-1)+ma}{ma} + \varepsilon_n(ma, mk)$$

$$= \frac{1}{m} \left[ C(a, k) + \frac{1}{k} \ln \frac{k(n-1) + a}{a} + \varepsilon_n(a, k) \right].$$

于是  $C(ma, mk) + \varepsilon_n(ma, mk)$

$$= \frac{1}{m} C(a, k) + \frac{1}{m} \varepsilon_n(a, k).$$

令  $n \rightarrow \infty$  并取极限, 命题即得证.

**命题2** 设  $b$  为正整数, 则

$$C(a + bk, k) = C(a, k) + \frac{1}{k} \ln \frac{a + bk}{a} - \left[ \frac{1}{a} + \frac{1}{k+a} + \frac{1}{2k+a} + \cdots + \frac{1}{(b-1)k+a} \right].$$

**证明** 根据定理2, 有

$$\begin{aligned} & \frac{1}{a} + \frac{1}{k+a} + \frac{1}{2k+a} + \cdots + \frac{1}{(b-1)k+a} \\ & - \left[ \frac{1}{kn+a} + \frac{1}{k(n+1)+a} + \cdots + \frac{1}{k(n-1+b)+a} \right] \\ & = H_n(a, k) - H_n(a + bk, k) \\ & = C(a, k) - C(a + bk, k) + \frac{1}{k} \ln \frac{nk - k + a}{nk - k + bk + a} \\ & \quad + \frac{1}{k} \ln \frac{a + bk}{a} + \varepsilon_n(a, k) - \varepsilon_n(a + bk, k). \end{aligned}$$

令  $m \rightarrow \infty$  并取极限, 命题即得证.

**定理3** 设  $A$  为非负整数,  $a, b, B, k$  都是正整数, 且  $b \geq 2$ , 则

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[ \frac{1}{k(Bn + A) + a} + \frac{1}{k(Bn + B + A) + a} \right. \\ & \quad \left. + \frac{1}{k(Bn + 2B + A) + a} + \cdots + \frac{1}{k(bBn - B + A) + a} \right] \\ & = \frac{1}{Bk} \ln b. \end{aligned}$$

**证明** 根据定理 2, 有

$$\begin{aligned} & \frac{1}{k(Bn+A)+a} + \frac{1}{k(Bn+B+A)+a} \\ & + \frac{1}{k(Bn+2B+A)+a} + \cdots + \frac{1}{k(bBn-B+A)+a} \\ & = H_{bn}(a+Ak, Bk) - H_n(a+Ak, Bk) \\ & = \frac{1}{Bk} \ln \frac{k(bBn-B+A)+a}{k(An-B+B)+a} + \varepsilon_{bn}(a+Ak, Bk) \\ & \quad - \varepsilon_n(a+Ak, Bk). \end{aligned}$$

令  $n \rightarrow \infty$  并取极限, 定理即得证.

令  $A=0, B=1$ , 则有

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[ \frac{1}{kn+a} + \frac{1}{k(n+1)+a} + \frac{1}{k(n+2)+a} \right. \\ \left. + \cdots + \frac{1}{k(bn-1)+a} \right] = \frac{1}{k} \ln b. \end{aligned}$$

于此令  $a=1, b=2, k=1$ , 即得通常所见的极限

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \right) = \ln 2.$$

**定理4** 设  $m$  为正整数, 且  $m \geq 2$ , 则

$$\begin{aligned} & C(a, mk) + C(a+k, mk) + C(a+2k, mk) \\ & + \cdots + C(a+mk-k, mk) - C(a, k) \\ & = \frac{1}{mk} \ln \frac{(a+k)(a+2k) \cdots (a+mk-k)}{a^{m-1}}. \end{aligned}$$

**证明** 根据定理 2, 有

$$\begin{aligned} & \frac{1}{kn+a} + \frac{1}{k(n+1)+a} + \frac{1}{k(n+2)+a} + \cdots \\ & + \frac{1}{k(mn-1)+a} \end{aligned}$$

$$\begin{aligned}
&= H_n(a, mk) + H_n(a+k, mk) + H_n(a+2k, mk) \\
&\quad + \cdots + H_n(a+mk-k, mk) - H_n(a, k) \\
&= C(a, mk) + C(a+k, mk) + C(a+2k, mk) \\
&\quad + \cdots + C(a+mk-k, mk) - C(a, k) \\
&\quad + \frac{1}{mk} \ln \frac{(mnk - mk + a) \cdots (mnk - k + a)}{(nk - k + a)^m} \\
&\quad - \frac{1}{mk} \ln \frac{a(a+k)(a+2k) \cdots (a+mk-k)}{a^m} \\
&\quad + \varepsilon_n(a, mk) + \varepsilon_n(a+k, mk) + \varepsilon_n(a+2k, mk) \\
&\quad + \cdots + \varepsilon_n(a+mk-k, mk) - \varepsilon_n(a, k).
\end{aligned}$$

注意到定理 3, 令  $n \rightarrow \infty$  并取极限, 定理即得证.



## 二、常数 $C(a, k)$ 与 $A(a, k)$ 的关系

记  $A(a, k) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{k(n-1) + a}$ .

**定理5**  $2C(a, 2k) - C(a, k) = A(a, k)$ .

**证明** 根据定理2, 有

$$\begin{aligned} & \sum_{i=1}^n \left[ \frac{1}{k(2i-2) + a} - \frac{1}{k(2i-1) + a} \right] \\ &= H_n(a, 2k) - H_n(a+k, 2k) \\ &= C(a, 2k) - C(a+k, 2k) + \frac{1}{2k} \ln \frac{2nk - 2k + a}{2nk - k + a} \\ & \quad + \frac{1}{2k} \ln \frac{a+k}{a} + \varepsilon_n(a, 2k) - \varepsilon_n(a+k, 2k). \end{aligned}$$

令  $n \rightarrow \infty$  并取极限即得

$$\begin{aligned} A(a, k) &= \sum_{n=1}^{\infty} \left[ \frac{1}{k(2n-2) + a} - \frac{1}{k(2n-1) + a} \right] \\ &= C(a, 2k) - C(a+k, 2k) + \frac{1}{2k} \ln \frac{a+k}{a}. \end{aligned} \quad (1)$$

于定理4 令  $m=2$ , 则有

$$\begin{aligned} & C(a, 2k) + C(a+k, 2k) - C(a, k) \\ &= \frac{1}{2k} \ln \frac{a+k}{a}, \end{aligned} \quad (2)$$

$$2C(a, 2k) - C(a, k)$$

$$= C(a, 2k) - C(a+k, 2k) + \frac{1}{2k} \ln \frac{a+k}{a}.$$

于是  $2C(a, 2k) - C(a, k) = A(a, k).$

例 求证  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \ln 2.$

证明 由(2)得

$$\begin{aligned} & C(a, 2k) - C(a+k, 2k) + \frac{1}{2k} \ln \frac{a+k}{a} \\ &= C(a, k) - 2C(a+k, 2k) + \frac{1}{k} \ln \frac{a+k}{a}. \end{aligned}$$

于是由(1)得

$$\begin{aligned} & A(a, k) \\ &= C(a, k) - 2C(a+k, 2k) + \frac{1}{k} \ln \frac{a+k}{a}. \end{aligned}$$

根据命题 1,  $2C(2, 2) = C(1, 1).$  于上式令  $a=1, k=1$  即得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = C(1, 1) - 2C(2, 2) + \ln 2 = \ln 2.$$

已求得  $A(1, 1) = \ln 2$ , 应用积分方法容易求得

$$A(1, 2) = \int_0^1 \frac{dx}{x^2+1} = \frac{\pi}{4}.$$

仿此求得

$$A(1, 3) = \frac{\pi}{3\sqrt{3}} + \frac{1}{3} \ln 2;$$

$$A(2, 3) = \frac{\pi}{3\sqrt{3}} - \frac{1}{3} \ln 2;$$

$$A(5, 3) = -A(2, 3) + \frac{1}{2} = -\frac{\pi}{3\sqrt{3}} + \frac{1}{3} \ln 2 + \frac{1}{2};$$

$$A(1, 4) = \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1);$$

$$A(3,4) = \frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1);$$

$$A(5,4) = -A(1,4) + 1$$

$$= -\frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1) + 1;$$

$$A(7,4) = -A(3,4) + \frac{1}{3}$$

$$= -\frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1) + \frac{1}{3};$$

$$A(1,6) = \frac{\pi}{6} + \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3});$$

$$A(5,6) = \frac{\pi}{6} - \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3});$$

$$A(7,6) = -A(1,6) + 1$$

$$= -\frac{\pi}{6} - \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}) + 1;$$

$$A(11,6) = -A(5,6) + \frac{1}{5}$$

$$= -\frac{\pi}{6} + \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{5}.$$

**定理6**  $2C(1,2) = C(1,1) + \ln 2;$

$$3C(1,3) = C(1,1) + \frac{\pi}{2\sqrt{3}} + \frac{1}{2} \ln 3;$$

$$3C(2,3) = C(1,1) - \frac{\pi}{2\sqrt{3}} + \frac{1}{2} \ln 12;$$

$$4C(1,4) = C(1,1) + \frac{\pi}{2} + \ln 2;$$

$$4C(3,4) = C(1,1) - \frac{\pi}{2} + \ln 6;$$

$$6C(1,6) = C(1,1) + \frac{\pi\sqrt{3}}{2} + \frac{1}{2} \ln 12,$$

$$6C(5,6) = C(1,1) - \frac{\pi\sqrt{3}}{2} + \frac{1}{2} \ln 300,$$

$$8C(1,8) = C(1,1) + \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 2 \\ + \sqrt{2} \ln (\sqrt{2} + 1);$$

$$8C(3,8) = C(1,1) + \frac{\pi}{\sqrt{2}} - \frac{\pi}{2} + \ln 6 \\ - \sqrt{2} \ln (\sqrt{2} + 1);$$

$$8C(5,8) = C(1,1) - \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 10 \\ - \sqrt{2} \ln (\sqrt{2} + 1);$$

$$8C(7,8) = C(1,1) - \frac{\pi}{\sqrt{2}} - \frac{\pi}{2} + \ln 14 \\ + \sqrt{2} \ln (\sqrt{2} + 1);$$

$$12C(1,12) = C(1,1) + \frac{\pi\sqrt{3}}{2} + \pi + \frac{1}{2} \ln 12 \\ + \sqrt{3} \ln (2 + \sqrt{3});$$

$$12C(5,12) = C(1,1) - \frac{\pi\sqrt{3}}{2} + \pi + \frac{1}{2} \ln 300 \\ - \sqrt{3} \ln (2 + \sqrt{3});$$

$$12C(7,12) = C(1,1) + \frac{\pi\sqrt{3}}{2} - \pi + \frac{1}{2} \ln 7^2 \cdot 12 \\ - \sqrt{3} \ln (2 + \sqrt{3});$$

$$12C(11,12) = C(1,1) - \frac{\pi\sqrt{3}}{2} - \pi + \frac{1}{2} \ln 11^2 \cdot 12 \\ + \sqrt{3} \ln (2 + \sqrt{3}).$$

**证明** 于定理 5 令  $a=1, k=1$  即得

$$2C(1,2) = C(1,1) + A(1,1) = C(1,1) + \ln 2.$$

于定理 5 令  $a=2, k=3$  即得

$$C(1,3) - C(2,3) = A(2,3) = \frac{\pi}{3\sqrt{3}} - \frac{1}{3} \ln 2.$$

于定理 4 令  $a=1, k=1, m=3$  即得

$$C(1,3) + C(2,3) + \frac{1}{3} C(1,1) - C(1,1) = \frac{1}{3} \ln 6.$$

由此即可求得

$$3C(1,3) = C(1,1) + \frac{\pi}{2\sqrt{3}} + \frac{1}{2} \ln 3,$$

$$3C(2,3) = C(1,1) - \frac{\pi}{2\sqrt{3}} + \frac{1}{2} \ln 12.$$

于定理 5 令  $a=1, k=2$  即得

$$2C(1,4) = C(1,2) + A(1,2),$$

于是  $4C(1,4) = 2C(1,2) + 2A(1,2)$

$$= C(1,1) + \frac{\pi}{2} + \ln 2.$$

于(1)令  $a=1, k=2$  即得

$$C(1,4) - C(3,4) + \frac{1}{4} \ln 3 = A(1,2) = \frac{\pi}{4},$$

于是  $4C(3,4) = 4C(1,4) - \pi + \ln 3 = C(1,1) - \frac{\pi}{2} + \ln 6.$

于定理 5 令  $a=1, k=3$  即得

$$2C(1,6) - C(1,3) = A(1,3) = \frac{\pi}{3\sqrt{3}} + \frac{1}{3} \ln 2,$$

于是  $6C(1,6) = 3C(1,3) + \frac{\pi}{\sqrt{3}} + \ln 2$

$$= C(1,1) + \frac{\pi\sqrt{3}}{2} + \frac{1}{2} \ln 12.$$

于定理 5 令  $a=5$ ,  $k=3$  即得

$$\begin{aligned} 2C(5,6) - C(5,3) \\ = A(5,3) = -\frac{\pi}{3\sqrt{3}} + \frac{1}{3} \ln 2 + \frac{1}{2}. \end{aligned}$$

于命题 2 令  $a=2$ ,  $b=1$ ,  $k=3$  即得

$$C(5,3) = C(2,3) + \frac{1}{3} \ln \frac{5}{2} - \frac{1}{2}.$$

于是  $2C(5,6) = C(2,3) - \frac{\pi}{3\sqrt{3}} + \frac{1}{3} \ln 5,$

$$\begin{aligned} 6C(5,6) &= 3C(2,3) - \frac{\pi}{\sqrt{3}} + \ln 5 \\ &= C(1,1) - \frac{\pi\sqrt{3}}{2} + \frac{1}{2} \ln 300. \end{aligned}$$

于定理 5 令  $a=1$ ,  $k=4$  即得

$$\begin{aligned} 2C(1,8) - C(1,4) \\ = A(1,4) = \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

于是  $8C(1,8) = 4C(1,4) + \frac{\pi}{\sqrt{2}} + \sqrt{2} \ln(\sqrt{2}+1)$

$$\begin{aligned} &= C(1,1) + \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 2 \\ &\quad + \sqrt{2} \ln(\sqrt{2}+1). \end{aligned}$$

于定理 5 令  $a=3$ ,  $k=4$  即得

$$\begin{aligned} 2C(3,8) - C(3,4) &= A(3,4) \\ &= \frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

$$\begin{aligned}
 \text{于是 } 8C(3,8) &= 4C(3,4) + \frac{\pi}{\sqrt{2}} - \sqrt{2} \ln(\sqrt{2}+1) \\
 &= C(1,1) + \frac{\pi}{\sqrt{2}} - \frac{\pi}{2} + \ln 6 \\
 &\quad - \sqrt{2} \ln(\sqrt{2}+1).
 \end{aligned}$$

于定理 5 令  $a=5$ ,  $k=4$  即得

$$\begin{aligned}
 2C(5,8) - C(5,4) \\
 = A(5,4) = -\frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1) + 1.
 \end{aligned}$$

于命题 2 令  $a=1$ ,  $b=1$ ,  $k=4$  即得

$$C(5,4) = C(1,4) + \frac{1}{4} \ln 5 - 1.$$

$$\text{于是 } 2C(5,8) = C(1,4) - \frac{\pi}{4\sqrt{2}} + \frac{1}{4} \ln 5$$

$$- \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1),$$

$$8C(5,8) = 4C(1,4) - \frac{\pi}{\sqrt{2}} + \ln 5$$

$$- \sqrt{2} \ln(\sqrt{2}+1)$$

$$= C(1,1) - \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 10$$

$$- \sqrt{2} \ln(\sqrt{2}+1).$$

于定理 5 令  $a=7$ ,  $k=4$  即得

$$2C(7,8) - C(7,4)$$

$$= A(7,4) = -\frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{3}.$$

于命题 2 令  $a=3$ ,  $b=1$ ,  $k=4$  即得

$$C(7,4) = C(3,4) + \frac{1}{4} \ln \frac{7}{3} - \frac{1}{3}.$$

于是  $2C(7,8) = C(3,4) - \frac{\pi}{4\sqrt{2}} + \frac{1}{4} \ln \frac{7}{3}$   
 $+ \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1),$

$$8C(7,8) = 4C(3,4) - \frac{\pi}{\sqrt{2}} + \ln \frac{7}{3}$$

$$+ \sqrt{2} \ln(\sqrt{2} + 1)$$

$$= C(1,1) - \frac{\pi}{\sqrt{2}} - \frac{\pi}{2} + \ln 14$$

$$+ \sqrt{2} \ln(\sqrt{2} + 1).$$

于定理 5 令  $a=1$ ,  $k=6$  即得

$$2C(1,12) - C(1,6) = A(1,6)$$

$$= \frac{\pi}{6} + \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}),$$

于是  $12C(1,12) = 6C(1,6) + \pi + \sqrt{3} \ln(2 + \sqrt{3})$

$$= C(1,1) + \frac{\pi\sqrt{3}}{2} + \pi + \frac{1}{2} \ln 12$$

$$+ \sqrt{3} \ln(2 + \sqrt{3}).$$

于定理 5 令  $a=5$ ,  $k=6$  即得

$$2C(5,12) - C(5,6) = A(5,6)$$

$$= \frac{\pi}{6} - \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}),$$

于是  $12C(5,12) = 6C(5,6) + \pi - \sqrt{3} \ln(2 + \sqrt{3})$

$$= C(1,1) - \frac{\pi\sqrt{3}}{2} + \pi + \frac{1}{2} \ln 300$$



$$- \sqrt{3} \ln(2 + \sqrt{3}).$$

于定理 5 令  $a=7$ ,  $k=6$  即得

$$\begin{aligned} 2C(7,12) - C(7,6) &= A(7,6) \\ &= -\frac{\pi}{6} - \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}) + 1. \end{aligned}$$

于命题 2 令  $a=1$ ,  $b=1$ ,  $k=6$  即得

$$C(7,6) = C(1,6) + \frac{1}{6} \ln 7 - 1.$$

于是  $2C(7,12) = C(1,6) - \frac{\pi}{6} + \frac{1}{6} \ln 7$

$$- \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\begin{aligned} 12C(7,12) &= 6C(1,6) - \pi + \ln 7 - \sqrt{3} \ln(2 + \sqrt{3}) \\ &= C(1,1) + \frac{\pi\sqrt{3}}{2} - \pi + \frac{1}{2} \ln 7^2 \cdot 12 \\ &\quad - \sqrt{3} \ln(2 + \sqrt{3}). \end{aligned}$$

于定理 5 令  $a=11$ ,  $k=6$  即得

$$\begin{aligned} 2C(11,12) - C(11,6) \\ &= A(11,6) = -\frac{\pi}{6} + \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{5}. \end{aligned}$$

于命题 2 令  $a=5$ ,  $b=1$ ,  $k=6$  即得

$$C(11,6) = C(5,6) + \frac{1}{6} \ln \frac{11}{5} - \frac{1}{5}.$$

于是  $2C(11,12) = C(5,6) - \frac{\pi}{6} + \frac{1}{6} \ln \frac{11}{5}$

$$+ \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\begin{aligned}
12C(11,12) &= 6C(5,6) - \pi + \ln \frac{11}{5} \\
&\quad + \sqrt{3} \ln(2 + \sqrt{3}) \\
&= C(1,1) - \frac{\pi\sqrt{3}}{2} - \pi + \frac{1}{2} \ln 12 \cdot 11^2 \\
&\quad + \sqrt{3} \ln(2 + \sqrt{3}).
\end{aligned}$$

应用积分方法还可求得

$$\begin{aligned}
A(1,5) &= \frac{\pi}{25} \sqrt{\frac{5+\sqrt{5}}{2}} + \frac{2\pi}{25} \sqrt{\frac{5-\sqrt{5}}{2}} + \frac{1}{5} \ln 2 \\
&\quad + \frac{1}{2\sqrt{5}} \ln \frac{3+\sqrt{5}}{2} \\
&= 0.8883135726\dots;
\end{aligned}$$

$$\begin{aligned}
A(2,5) &= \frac{\pi}{5} \left( \frac{\sqrt{5}-1}{\sqrt{50+10\sqrt{5}}} - \frac{1}{5} \sqrt{\frac{5+\sqrt{5}}{2}} \right) \\
&\quad + \frac{2\pi}{5} \left( \frac{\sqrt{5}+1}{\sqrt{50-10\sqrt{5}}} - \frac{1}{5} \sqrt{\frac{5-\sqrt{5}}{2}} \right) \\
&\quad - \frac{1}{5} \ln 2 + \frac{1}{2\sqrt{5}} \ln \frac{3+\sqrt{5}}{2} \\
&= 0.4069016342\dots;
\end{aligned}$$

$$\begin{aligned}
A(3,5) &= \frac{\pi}{5} \left( \frac{1}{5} \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+3}{\sqrt{50+10\sqrt{5}}} \right) \\
&\quad + \frac{2\pi}{5} \left( \frac{1}{5} \sqrt{\frac{5-\sqrt{5}}{2}} + \frac{3-\sqrt{5}}{\sqrt{50-10\sqrt{5}}} \right) \\
&\quad + \frac{1}{5} \ln 2 - \frac{1}{2\sqrt{5}} \ln \frac{3+\sqrt{5}}{2} \\
&= 0.2537515655\dots;
\end{aligned}$$

$$\begin{aligned}
A(4,5) &= \frac{\pi}{5} \left( \frac{2\sqrt{5}+2}{\sqrt{50+10\sqrt{5}}} - \frac{1}{5} \sqrt{\frac{5+\sqrt{5}}{2}} \right) \\
&\quad + \frac{2\pi}{5} \left( \frac{2\sqrt{5}-2}{\sqrt{50-10\sqrt{5}}} - \frac{1}{5} \sqrt{\frac{5-\sqrt{5}}{2}} \right) \\
&\quad - \frac{1}{5} \ln 2 - \frac{1}{2\sqrt{5}} \ln \frac{3+\sqrt{5}}{2} \\
&= 0.1806457593\dots;
\end{aligned}$$

$$A(6,5) = -A(1,5) + 1; \quad A(7,5) = -A(2,5) + \frac{1}{2},$$

$$A(8,5) = -A(3,5) + \frac{1}{3},$$

$$A(9,5) = -A(4,5) + \frac{1}{4}.$$

$$\begin{aligned}
A(1,8) &= \frac{\sqrt{2}-1}{8\sqrt{4-2\sqrt{2}}} \ln \frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}} \\
&\quad + \frac{\sqrt{2}+1}{8\sqrt{4+2\sqrt{2}}} \ln \frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}} \\
&\quad + \frac{\pi}{2} \left( \frac{1}{4\sqrt{4-2\sqrt{2}}} + \frac{1}{4\sqrt{4+2\sqrt{2}}} \right) \\
&= 0.9246517057\dots;
\end{aligned}$$

$$A(3,8) = -\frac{1}{8\sqrt{4-2\sqrt{2}}} \ln \frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$$

$$\begin{aligned}
& + \frac{1}{8\sqrt{4+2\sqrt{2}}} \ln \frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}} \\
& + \frac{\pi}{2} \left( \frac{1}{4\sqrt{4-2\sqrt{2}}} - \frac{1}{4\sqrt{4+2\sqrt{2}}} \right) \\
& = 0.2738982192\dots;
\end{aligned}$$

$$\begin{aligned}
A(5,8) &= \frac{1}{8\sqrt{4-2\sqrt{2}}} \ln \frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}} \\
& - \frac{1}{8\sqrt{4+2\sqrt{2}}} \ln \frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}} \\
& + \frac{\pi}{2} \left( \frac{1}{4\sqrt{4-2\sqrt{2}}} - \frac{1}{4\sqrt{4+2\sqrt{2}}} \right) \\
& = 0.1511562038\dots;
\end{aligned}$$

$$\begin{aligned}
A(7,8) &= -\frac{\sqrt{2}-1}{8\sqrt{4-2\sqrt{2}}} \ln \frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}} \\
& - \frac{\sqrt{2}+1}{8\sqrt{4+2\sqrt{2}}} \ln \frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}} \\
& + \frac{\pi}{2} \left( \frac{\sqrt{2}+1}{4\sqrt{4+2\sqrt{2}}} + \frac{\sqrt{2}-1}{4\sqrt{4-2\sqrt{2}}} \right) \\
& = 0.1015204471\dots;
\end{aligned}$$

$$A(9,8) = -A(1,8) + 1,$$

$$A(11,8) = -A(3,8) + \frac{1}{3},$$

$$A(13,8) = -A(5,8) + \frac{1}{5},$$

$$A(15,8) = -A(7,8) + \frac{1}{7}.$$

**定理7**  $5C(1,5) = C(1,1) + \frac{5}{4} A(1,5) + \frac{5}{2} A(2,5)$   
 $+ \frac{5}{4} A(4,5) + \frac{1}{4} \ln 20;$

$$5C(2,5) = C(1,1) + \frac{5}{4} A(1,5) - \frac{5}{2} A(2,5)$$

$$+ \frac{5}{4} A(4,5) + \frac{1}{4} \ln 20;$$

$$5C(3,5) = C(1,5) - \frac{15}{4} A(1,5) + \frac{5}{2} A(2,5)$$

$$+ \frac{5}{4} A(4,5) + \frac{1}{4} \ln 20 + \ln 6;$$

$$5C(4,5) = C(1,1) + \frac{5}{4} A(1,5) - \frac{5}{2} A(2,5)$$

$$- \frac{15}{4} A(4,5) + \frac{1}{4} \ln 20;$$

$$10C(1,10) = C(1,1) + \frac{25}{4} A(1,5) + \frac{5}{2} A(2,5)$$

$$+ \frac{5}{4} A(4,5) + \frac{1}{4} \ln 20;$$

$$10C(3,10) = C(1,1) - \frac{15}{4} A(1,5) + \frac{5}{2} A(2,5)$$

$$+ 5A(3,5) + \frac{5}{4} A(4,5) + \frac{1}{4} \ln 20 + \ln 6;$$

$$10C(7,10) = C(1,1) + \frac{5}{4} A(1,5) - \frac{15}{2} A(2,5) \\ + \frac{5}{4} A(4,5) + \frac{1}{4} \ln 20 + \ln \frac{7}{2};$$

$$10C(9,10) = C(1,1) + \frac{5}{4} A(1,5) - \frac{5}{2} A(2,5) \\ - \frac{35}{4} A(4,5) + \frac{1}{4} \ln 20 + \ln \frac{9}{4}.$$

**证明** 于(1) 令  $a=1, k=5$ , 则有

$$C(1,10) - \frac{1}{2} C(3,5) + \frac{1}{10} \ln 6 = A(1,5),$$

$$2C(1,10) - C(3,5) + \frac{1}{5} \ln 6 = 2A(1,5).$$

于定理 5 令  $a=1, k=5$ , 则有

$$2C(1,10) - C(1,5) = A(1,5).$$

由此得出

$$C(1,5) - C(3,5) + \frac{1}{5} \ln 6 = A(1,5).$$

于定理 5 令  $a=2, k=5$ , 则有

$$C(1,5) - C(2,5) = A(2,5).$$

于定理 5 令  $a=4, k=5$ , 则有

$$C(2,5) - C(4,5) = A(4,5).$$

于定理 4 令  $a=1, k=1, m=5$ , 则有

$$C(1,5) + C(2,5) + C(3,5) + C(4,5) \\ = \frac{4}{5} C(1,1) + \frac{1}{5} \ln 120.$$

由以上四个等式即可解出  $C(1,5)$ 、 $C(2,5)$ 、 $C(3,5)$  以及  $C(4,5)$ 。

于定理 5 令  $a=1$ ,  $k=5$ , 则有

$$2C(1,10) = C(1,5) + A(1,5),$$

于是  $10C(1,10) = 5C(1,5) + 5A(1,5)$

$$\begin{aligned} &= C(1,1) + \frac{25}{4} A(1,5) + \frac{5}{2} A(2,5) \\ &\quad + \frac{5}{4} A(4,5) + \frac{1}{4} \ln 20. \end{aligned}$$

于定理 5 令  $a=3$ ,  $k=5$ , 则有

$$2C(3,10) = C(3,5) + A(3,5),$$

于是  $10C(3,10) = 5C(3,5) + 5A(3,5)$

$$\begin{aligned} &= C(1,1) - \frac{15}{4} A(1,5) + \frac{5}{2} A(2,5) \\ &\quad + 5A(3,5) + \frac{5}{4} A(4,5) + \frac{1}{4} \ln 20 \\ &\quad + \ln 6. \end{aligned}$$

于定理 5 令  $a=7$ ,  $k=5$ , 则有

$$2C(7,10) = C(7,5) + A(7,5)$$

$$= C(7,5) - A(2,5) + \frac{1}{2}.$$

于命题 2 令  $a=2$ ,  $b=1$ ,  $k=5$ , 则有

$$C(7,5) = C(2,5) + \frac{1}{5} \ln \frac{7}{2} - \frac{1}{2}.$$

于是  $2C(7,10) = C(2,5) - A(2,5) + \frac{1}{5} \ln \frac{7}{2},$

$$\begin{aligned}
10C(7,10) &= 5C(2,5) - 5A(2,5) + \ln \frac{7}{2} \\
&= C(1,1) + \frac{5}{4} A(1,5) - \frac{15}{2} A(2,5) \\
&\quad + \frac{5}{4} A(4,5) + \frac{1}{4} \ln 20 + \ln \frac{7}{2}.
\end{aligned}$$

于定理 5 令  $a=9$ ,  $k=5$ , 则有

$$\begin{aligned}
2C(9,10) &= C(9,5) + A(9,5) \\
&= C(9,5) - A(4,5) + \frac{1}{4}.
\end{aligned}$$

于命题 2 令  $a=4$ ,  $b=1$ ,  $k=5$ , 则有

$$C(9,5) = C(4,5) + \frac{1}{5} \ln \frac{9}{4} - \frac{1}{4}.$$

于是  $2C(9,10) = C(4,5) - A(4,5) + \frac{1}{5} \ln \frac{9}{4}$ ,

$$\begin{aligned}
10C(9,10) &= 5C(4,5) - 5A(4,5) + \ln \frac{9}{4} \\
&= C(1,1) + \frac{5}{4} A(1,5) - \frac{5}{2} A(2,5) \\
&\quad - \frac{35}{4} A(4,5) + \frac{1}{4} \ln 20 + \frac{9}{4}.
\end{aligned}$$

**定理 8**  $16C(1,16) = C(1,1) + \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 2$   
 $+ \sqrt{2} \ln (\sqrt{2} + 1) + 8A(1,8),$

$$\begin{aligned}
16C(3,16) &= C(1,1) + \frac{\pi}{\sqrt{2}} - \frac{\pi}{2} + \ln 6 \\
&\quad - \sqrt{2} \ln (\sqrt{2} + 1) + 8A(3,8),
\end{aligned}$$

$$16C(5,16) = C(1,1) - \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 10$$



$$\begin{aligned}
& -\sqrt{2} \ln(\sqrt{2}+1) + 8A(5,8), \\
16C(7,16) &= C(1,1) - \frac{\pi}{\sqrt{2}} - \frac{\pi}{2} + \ln 14 \\
& + \sqrt{2} \ln(\sqrt{2}+1) + 8A(7,8), \\
16C(9,16) &= C(1,1) + \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 18 \\
& + \sqrt{2} \ln(\sqrt{2}+1) - 8A(1,8), \\
16C(11,16) &= C(1,1) + \frac{\pi}{\sqrt{2}} - \frac{\pi}{2} + \ln 22 \\
& - \sqrt{2} \ln(\sqrt{2}+1) - 8A(3,8), \\
16C(13,16) &= C(1,1) - \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 26 \\
& - \sqrt{2} \ln(\sqrt{2}+1) - 8A(5,8), \\
16C(15,16) &= C(1,1) - \frac{\pi}{\sqrt{2}} - \frac{\pi}{2} + \ln 30 \\
& + \sqrt{2} \ln(\sqrt{2}+1) - 8A(7,8).
\end{aligned}$$

**证明** 于定理 5 令  $a=1$ ,  $k=8$ , 则有

$$2C(1,16) - C(1,8) = A(1,8).$$

于是  $16C(1,16) = 8C(1,8) + 8A(1,8)$

$$\begin{aligned}
&= C(1,1) + \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 2 \\
&+ \sqrt{2} \ln(\sqrt{2}+1) + 8A(1,8).
\end{aligned}$$

仿此可得  $16C(3,16)$ 、 $16C(5,16)$  以及  $16C(7,16)$ 。

于定理 5 令  $a=9$ ,  $k=8$ , 则有

$$\begin{aligned}
2C(9,16) &= C(9,8) + A(9,8) \\
&= C(9,8) - A(1,8) + 1.
\end{aligned}$$

于命题 2 令  $a=1$ ,  $b=1$ ,  $k=8$ , 则有

$$C(9,8) = C(1,8) + \frac{1}{8} \ln 9 - 1.$$

于是  $2C(9,16) = C(1,8) + \frac{1}{8} \ln 9 - A(1,8),$

$$16C(9,16) = 8C(1,8) + \ln 9 - 8A(1,8)$$

$$= C(1,1) + \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} + \ln 18$$

$$+ \sqrt{2} \ln(\sqrt{2} + 1) - 8A(1,8).$$

仿此可得  $16C(11,16)$ 、 $16C(13,16)$  以及  $16C(15,16)$ .

应用定理 6、7、8 所给出的各个常数  $C(a,k)$ , 由以下的诸定理将推得各种类型的无穷级数之和.

### 三、无穷级数的求和

$$1. \sum_{n=1}^{\infty} \left[ \frac{1}{k(n-1)+a} - \frac{1}{k(n-1)+b} \right]$$

**定理9** 设  $a$ 、 $b$ 、 $k$  是正整数, 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(n-1)+a} - \frac{1}{k(n-1)+b} \right] \\ &= C(a, k) - C(b, k) + \frac{1}{k} \ln \frac{b}{a}. \end{aligned}$$

**证明** 根据定理2, 有

$$\begin{aligned} & \sum_{i=1}^n \left[ \frac{1}{k(i-1)+a} - \frac{1}{k(i-1)+b} \right] \\ &= C(a, k) - C(b, k) + \frac{1}{k} \ln \frac{nk - k + a}{nk - k + b} \\ & \quad + \frac{1}{k} \ln \frac{b}{a} + \varepsilon_n(a, k) - \varepsilon_n(b, k). \end{aligned}$$

令  $n \rightarrow \infty$  并取极限, 定理即得证.

**例1** 求和  $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-2)(4n-1)4n(4n+1)}$ .

**解** 于定理9 令  $a=1$ ,  $b=2$ ,  $k=4$ , 则有

$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-2)} = \sum_{n=1}^{\infty} \left( \frac{1}{4n-3} - \frac{1}{4n-2} \right)$$

$$= C(1, 4) - \frac{1}{2} C(1, 2) + \frac{1}{4} \ln 2.$$

根据定理 6, 有

$$C(1, 4) - \frac{1}{2} C(1, 2) = \frac{\pi}{8}.$$

于是 
$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-2)} = \frac{\pi}{8} + \frac{1}{4} \ln 2.$$

仿此求得

$$\sum_{n=1}^{\infty} \frac{1}{(4n-2)(4n-1)} = \frac{\pi}{8} - \frac{1}{4} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)4n} = -\frac{\pi}{8} + \frac{3}{4} \ln 2.$$

于定理 9 令  $a=4$ ,  $b=5$ ,  $k=4$ , 则有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{4n(4n+1)} &= \sum_{n=1}^{\infty} \left( \frac{1}{4n} - \frac{1}{4n+1} \right) \\ &= \frac{1}{4} C(1, 1) - C(5, 4) + \frac{1}{4} \ln \frac{5}{2^2}. \end{aligned}$$

于命题 2 令  $a=1$ ,  $b=1$ ,  $k=4$ , 则有

$$C(5, 4) = C(1, 4) + \frac{1}{4} \ln 5 - 1.$$

于是 
$$\sum_{n=1}^{\infty} \frac{1}{4n(4n+1)} = \frac{1}{4} C(1, 1) - C(1, 4) - \frac{1}{2} \ln 2 + 1.$$

根据定理 6, 有

$$\frac{1}{4} C(1, 1) - C(1, 4) = -\frac{\pi}{8} - \frac{1}{4} \ln 2.$$

于是 
$$\sum_{n=1}^{\infty} \frac{1}{4n(4n+1)} = -\frac{\pi}{8} - \frac{3}{4} \ln 2 + 1.$$

从上面所得的结果即可顺次求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-2)(4n-1)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(4n-2)} - \frac{1}{(4n-2)(4n-1)} \right] \\
&= \frac{1}{4} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-2)(4n-1)4n} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-2)(4n-1)} - \frac{1}{(4n-1)4n} \right] \\
&= \frac{\pi}{8} - \frac{1}{2} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-1)4n(4n+1)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)4n} - \frac{1}{4n(4n+1)} \right] \\
&= \frac{3}{4} \ln 2 - \frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-2)(4n-1)4n} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(4n-2)(4n-1)} - \frac{1}{(4n-2)(4n-1)4n} \right] \\
&= -\frac{\pi}{24} + \frac{1}{4} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-2)(4n-1)4n(4n+1)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-2)(4n-1)4n} - \frac{1}{(4n-1)4n(4n+1)} \right]
\end{aligned}$$

$$= \frac{\pi}{24} - \frac{5}{12} \ln 2 + \frac{1}{6}.$$

于是

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-2)(4n-1)4n(4n+1)} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(4n-2)(4n-1)4n} \right. \\ & \quad \left. - \frac{1}{(4n-2)(4n-1)4n(4n+1)} \right] \\ &= -\frac{\pi}{48} + \frac{1}{6} \ln 2 - \frac{1}{24}. \end{aligned}$$

例2 求和  $\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-4)(8n-1)}.$

解 于定理9 令  $a=1, b=4, k=8$ , 即可推得

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-4)} &= \frac{1}{3} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} - \frac{1}{8n-4} \right) \\ &= \frac{\pi}{24\sqrt{2}} + \frac{\pi}{48} + \frac{1}{12} \ln 2 + \frac{1}{12\sqrt{2}} \ln(\sqrt{2}+1). \end{aligned}$$

令  $a=4, b=7, k=8$ , 即可推得

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(8n-4)(8n-1)} &= \frac{1}{3} \sum_{n=1}^{\infty} \left( \frac{1}{8n-4} - \frac{1}{8n-1} \right) \\ &= \frac{\pi}{24\sqrt{2}} + \frac{\pi}{48} - \frac{1}{12} \ln 2 - \frac{1}{12\sqrt{2}} \ln(\sqrt{2}+1). \end{aligned}$$

于是

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-4)(8n-1)} \\ &= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-4)} - \frac{1}{(8n-4)(8n-1)} \right] \end{aligned}$$

$$= \frac{1}{36} \ln 2 + \frac{1}{36\sqrt{2}} \ln(\sqrt{2} + 1).$$

仿此求得一系列的级数之和:

$$(一) \quad \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)}$$

$$= \frac{\pi}{24\sqrt{3}} + \frac{1}{3} \ln 2 - \frac{1}{8} \ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-4)(6n-3)(6n-2)}$$

$$= \frac{1}{4} \ln 3 - \frac{1}{3} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-3)(6n-2)(6n-1)}$$

$$= -\frac{\pi}{24\sqrt{3}} + \frac{1}{3} \ln 2 - \frac{1}{8} \ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)6n}$$

$$= \frac{5\pi}{24\sqrt{3}} - \frac{1}{3} \ln 2 - \frac{1}{8} \ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-1)6n(6n+1)}$$

$$= \frac{1}{4} \ln 3 + \frac{1}{3} \ln 2 - \frac{1}{2}.$$

由此顺次求得

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)3n}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(6n-3)} + \frac{1}{(6n-2)(6n-1)6n} \right]$$

$$= \frac{\pi}{4\sqrt{3}} - \frac{1}{4} \ln 3,$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)3n} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(6n-3)} - \frac{1}{(6n-2)(6n-1)6n} \right] \\
&= -\frac{\pi}{6\sqrt{3}} + \frac{2}{3} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(6n-3)} \right. \\
&\quad \left. - \frac{1}{(6n-4)(6n-3)(6n-2)} \right] \\
&= \frac{\pi}{72\sqrt{3}} - \frac{1}{8} \ln 3 + \frac{2}{9} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)} \\
&= \frac{\pi}{72\sqrt{3}} + \frac{1}{8} \ln 3 - \frac{2}{9} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-3)(6n-2)(6n-1)6n} \\
&= -\frac{\pi}{12\sqrt{3}} + \frac{2}{9} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)6n(6n+1)} \\
&= \frac{5\pi}{72\sqrt{3}} - \frac{2}{9} \ln 2 - \frac{1}{8} \ln 3 + \frac{1}{6}.
\end{aligned}$$



$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)3n(3n+1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)} \right. \\ \left. + \frac{1}{(6n-2)(6n-1)6n(6n+1)} \right]$$

$$= \frac{\pi}{12\sqrt{3}} - \frac{1}{4} \ln 3 + \frac{1}{6},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)3n(3n+1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)} \right. \\ \left. - \frac{1}{(6n-2)(6n-1)6n(6n+1)} \right]$$

$$= -\frac{\pi}{18\sqrt{3}} + \frac{4}{9} \ln 2 - \frac{1}{6},$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)} \right. \\ \left. - \frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)} \right]$$

$$= \frac{1}{9} \ln 2 - \frac{1}{16} \ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)6n}$$

$$= \frac{7\pi}{288\sqrt{3}} + \frac{1}{32} \ln 3 - \frac{1}{9} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-3)(6n-2)(6n-1)6n(6n+1)}$$

$$= -\frac{11\pi}{288\sqrt{3}} + \frac{1}{9} \ln 2 + \frac{1}{32} \ln 3 - \frac{1}{24},$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)6n}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$- \frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)6n} \Big]$$

$$= -\frac{7\pi}{1440\sqrt{3}} + \frac{2}{45} \ln 2 - \frac{3}{160} \ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)6n(6n+1)}$$

$$= \frac{\pi}{80\sqrt{3}} - \frac{2}{45} \ln 2 + \frac{1}{120},$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)6n(6n+1)}$$

$$= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)6n} \right.$$

$$\left. - \frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)6n(6n+1)} \right]$$

$$= -\frac{5\pi}{1728\sqrt{3}} + \frac{2}{135} \ln 2 - \frac{1}{320} \ln 3 - \frac{1}{720}.$$

$$(二) \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)}$$

$$= \frac{\pi}{8\sqrt{2}} - \frac{\pi}{16} + \frac{1}{8} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-6)(8n-5)(8n-4)}$$

$$= -\frac{\pi}{8\sqrt{2}} + \frac{3\pi}{32} - \frac{3}{16} \ln 2 + \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-5)(8n-4)(8n-3)}$$

$$= \frac{1}{4} \ln 2 - \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-4)(8n-3)(8n-2)}$$

$$= \frac{\pi}{8\sqrt{2}} - \frac{3\pi}{32} - \frac{3}{16} \ln 2 + \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-3)(8n-2)(8n-1)}$$

$$= -\frac{\pi}{8\sqrt{2}} + \frac{\pi}{16} + \frac{1}{8} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-2)(8n-1)8n}$$

$$= \frac{\pi}{8\sqrt{2}} + \frac{\pi}{32} - \frac{5}{16} \ln 2 - \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1).$$

由此顺次求得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(4n-3)(4n-2)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-6)(8n-5)} \right. \\ \left. - \frac{1}{(8n-3)(8n-2)(8n-1)} \right]$$

$$= \frac{\pi}{4\sqrt{2}} - \frac{\pi}{8},$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)}$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-6)(8n-5)} - \frac{1}{(8n-6)(8n-5)(8n-4)} \right]$$

$$= \frac{\pi}{12\sqrt{2}} - \frac{5\pi}{96} + \frac{5}{48} \ln 2 - \frac{1}{12\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$= -\frac{\pi}{24\sqrt{2}} + \frac{\pi}{32} - \frac{7}{48} \ln 2 + \frac{1}{6\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-5)(8n-4)(8n-3)(8n-2)}$$

$$= -\frac{\pi}{24\sqrt{2}} + \frac{\pi}{32} + \frac{7}{48} \ln 2 - \frac{1}{6\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$= \frac{\pi}{12\sqrt{2}} - \frac{5\pi}{96} - \frac{5}{48} \ln 2 + \frac{1}{12\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-3)(8n-2)(8n-1)8n}$$

$$= -\frac{\pi}{12\sqrt{2}} + \frac{\pi}{96} + \frac{7}{48} \ln 2 + \frac{1}{12\sqrt{2}} \ln(\sqrt{2} + 1).$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(4n-3)(4n-2)(4n-1)4n}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)} \right]$$

$$\begin{aligned}
& - \frac{1}{(8n-3)(8n-2)(8n-1)8n} \Big] \\
& = \frac{\pi}{6\sqrt{2}} - \frac{\pi}{16} - \frac{1}{24} \ln 2 - \frac{1}{6\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \\
& = \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)} \right. \\
& \quad \left. - \frac{1}{(8n-6)(8n-5)(8n-4)(8n-3)} \right] \\
& = \frac{\pi}{32\sqrt{2}} - \frac{\pi}{48} + \frac{1}{16} \ln 2 - \frac{1}{16\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)} \\
& = -\frac{7}{96} \ln 2 + \frac{1}{12\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-5)(8n-4)(8n-3)(8n-2)(8n-1)} \\
& = -\frac{\pi}{32\sqrt{2}} + \frac{\pi}{48} + \frac{1}{16} \ln 2 - \frac{1}{16\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-4)(8n-3)(8n-2)(8n-1)8n} \\
& = \frac{\pi}{24\sqrt{2}} - \frac{\pi}{64} - \frac{1}{16} \ln 2; \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)} \\
& = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)} \Big] \\
& = \frac{\pi}{160\sqrt{2}} - \frac{\pi}{240} + \frac{13}{480} \ln 2 - \frac{7}{240\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)} \\
& = \frac{\pi}{160\sqrt{2}} - \frac{\pi}{240} - \frac{13}{480} \ln 2 + \frac{7}{240\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-5)(8n-4)(8n-3)\cdots 8n} \\
& = -\frac{7\pi}{480\sqrt{2}} + \frac{7\pi}{960} + \frac{1}{40} \ln 2 - \frac{1}{80\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)} \\
& = \frac{13}{1440} \ln 2 - \frac{7}{720\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-6)(8n-5)(8n-4)\cdots 8n} \\
& = \frac{\pi}{288\sqrt{2}} - \frac{11\pi}{5760} - \frac{5}{576} \ln 2 \\
& \quad + \frac{1}{144\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)\cdots 8n} \\
& = -\frac{\pi}{2016\sqrt{2}} + \frac{11\pi}{40320} + \frac{17}{6720} \ln 2 \\
& \quad - \frac{1}{420\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

$$\begin{aligned}
(\text{三}) \quad & \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(8n-3)} \\
&= -\frac{\pi}{32\sqrt{2}} + \frac{\pi}{32} + \frac{1}{16\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-5)(8n-3)(8n-1)} \\
&= \frac{\pi}{32\sqrt{2}} - \frac{\pi}{32} + \frac{1}{16\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-3)(8n-1)(8n+1)} \\
&= \frac{\pi}{32\sqrt{2}} + \frac{\pi}{32} - \frac{1}{16\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{8}.
\end{aligned}$$

由此顺次求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(4n-3)(4n-1)(4n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-5)(8n-3)} \right. \\
& \quad \left. - \frac{1}{(8n-3)(8n-1)(8n+1)} \right] \\
&= -\frac{\pi}{16\sqrt{2}} + \frac{1}{8\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{8}, \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(8n-3)(8n-1)} \\
&= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-5)(8n-3)} \right. \\
& \quad \left. - \frac{1}{(8n-5)(8n-3)(8n-1)} \right]
\end{aligned}$$

$$= -\frac{\pi}{96\sqrt{2}} + \frac{\pi}{96},$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$= -\frac{\pi}{96} + \frac{1}{48\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{48},$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(8n-3)(8n-1)(8n+1)}$$

$$= \frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-5)(8n-3)(8n-1)} \right.$$

$$\left. - \frac{1}{(8n-5)(8n-3)(8n-1)(8n+1)} \right]$$

$$= -\frac{\pi}{768\sqrt{2}} + \frac{\pi}{384} - \frac{1}{384\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{384}.$$

$$(四) \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)}$$

$$= -\frac{\pi}{16\sqrt{3}} + \frac{\pi}{16} + \frac{1}{12} \ln 2 - \frac{1}{16} \ln 3$$

$$+ \frac{1}{8\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)}$$

$$= \frac{\pi}{12\sqrt{3}} - \frac{\pi}{24} + \frac{1}{8} \ln 3 - \frac{1}{6} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-8)(12n-7)}$$

$$= -\frac{5\pi}{48\sqrt{3}} + \frac{\pi}{16} + \frac{1}{4} \ln 2 - \frac{1}{16} \ln 3$$



$$-\frac{1}{8\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-8)(12n-7)(12n-6)} \\ &= \frac{7\pi}{48\sqrt{3}} - \frac{\pi}{12} - \frac{1}{16} \ln 3 - \frac{1}{6} \ln 2 \\ & \quad + \frac{1}{4\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-7)(12n-6)(12n-5)} \\ &= \frac{1}{8} \ln 3 + \frac{1}{12} \ln 2 - \frac{1}{4\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-6)(12n-5)(12n-4)} \\ &= -\frac{7\pi}{48\sqrt{3}} + \frac{\pi}{12} - \frac{1}{16} \ln 3 - \frac{1}{6} \ln 2 \\ & \quad + \frac{1}{4\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-4)(12n-3)} \\ &= \frac{5\pi}{48\sqrt{3}} - \frac{\pi}{16} - \frac{1}{16} \ln 3 + \frac{1}{4} \ln 2 \\ & \quad - \frac{1}{8\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-4)(12n-3)(12n-2)} \\ &= -\frac{\pi}{12\sqrt{3}} + \frac{\pi}{24} + \frac{1}{8} \ln 3 - \frac{1}{6} \ln 2, \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-3)(12n-2)(12n-1)}$$

$$= \frac{\pi}{16\sqrt{3}} - \frac{\pi}{16} + \frac{1}{12} \ln 2 - \frac{1}{16} \ln 3$$

$$+ \frac{1}{8\sqrt{3}} \ln (2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-2)(12n-1)12n}$$

$$= \frac{\pi}{16\sqrt{3}} + \frac{\pi}{12} - \frac{1}{6} \ln 2 - \frac{1}{16} \ln 3$$

$$- \frac{1}{4\sqrt{3}} \ln (2 + \sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-4)(6n-3)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)} \right.$$

$$\left. - \frac{1}{(12n-5)(12n-4)(12n-3)} \right]$$

$$= -\frac{\pi}{6\sqrt{3}} + \frac{\pi}{8} - \frac{1}{6} \ln 2 + \frac{1}{4\sqrt{3}} \ln (2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$= \frac{1}{3} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)} \right.$$

$$\left. - \frac{1}{(12n-10)(12n-9)(12n-8)} \right]$$

$$= -\frac{7\pi}{144\sqrt{3}} + \frac{5\pi}{144} + \frac{1}{12} \ln 2 - \frac{1}{16} \ln 3$$

$$+ \frac{1}{24\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)}$$

$$= \frac{\pi}{16\sqrt{3}} - \frac{5\pi}{144} + \frac{1}{16} \ln 3 - \frac{5}{36} \ln 2$$

$$+ \frac{1}{24\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$= -\frac{\pi}{12\sqrt{3}} + \frac{7\pi}{144} + \frac{5}{36} \ln 2 - \frac{1}{8\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$= \frac{7\pi}{144\sqrt{3}} - \frac{\pi}{36} - \frac{1}{16} \ln 3 - \frac{1}{12} \ln 2$$

$$+ \frac{1}{6\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$= \frac{7\pi}{144\sqrt{3}} - \frac{\pi}{36} + \frac{1}{16} \ln 3 + \frac{1}{12} \ln 2$$

$$- \frac{1}{6\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-6)(12n-5)(12n-4)(12n-3)}$$

$$= -\frac{\pi}{12\sqrt{3}} + \frac{7\pi}{144} - \frac{5}{36} \ln 2 + \frac{1}{8\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$= \frac{\pi}{16\sqrt{3}} - \frac{5\pi}{144} - \frac{1}{16} \ln 3 + \frac{5}{36} \ln 2$$

$$- \frac{1}{24\sqrt{3}} \ln (2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$= -\frac{7\pi}{144\sqrt{3}} + \frac{5\pi}{144} + \frac{1}{16} \ln 3 - \frac{1}{12} \ln 2$$

$$- \frac{1}{24\sqrt{3}} \ln (2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-3)(12n-2)(12n-1)12n}$$

$$= -\frac{7\pi}{144} + \frac{1}{12} \ln 2 + \frac{1}{8\sqrt{3}} \ln (2 + \sqrt{3}).$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)} \right.$$

$$\left. - \frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)} \right]$$

$$= -\frac{\pi}{9\sqrt{3}} + \frac{5\pi}{72} - \frac{1}{18} \ln 2 + \frac{1}{12\sqrt{3}} \ln (2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)} \right.$$

$$\left. - \frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)} \right]$$

$$= -\frac{1}{36\sqrt{3}} + \frac{5\pi}{288} - \frac{1}{32} \ln 3 + \frac{1}{18} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$= \frac{7\pi}{192\sqrt{3}} - \frac{\pi}{48} + \frac{1}{64} \ln 3 - \frac{5}{72} \ln 2$$

$$+ \frac{1}{24\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$= -\frac{19\pi}{576\sqrt{3}} + \frac{11\pi}{576} + \frac{1}{64} \ln 3 + \frac{1}{18} \ln 2$$

$$- \frac{7}{96\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-8)(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$= -\frac{1}{32} \ln 3 - \frac{1}{24} \ln 2 + \frac{1}{12\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-7)(12n-6)(12n-5)(12n-4)(12n-3)}$$

$$= \frac{19\pi}{576\sqrt{3}} - \frac{11\pi}{576} + \frac{1}{64} \ln 3 + \frac{1}{18} \ln 2$$

$$- \frac{7}{96\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-6)(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$= -\frac{7\pi}{192\sqrt{3}} + \frac{\pi}{48} + \frac{1}{64} \ln 3 - \frac{5}{72} \ln 2$$

$$+ \frac{1}{24\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$= \frac{\pi}{36\sqrt{3}} - \frac{5\pi}{288} - \frac{1}{32} \ln 3 + \frac{1}{18} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)12n}$$

$$= -\frac{7\pi}{576\sqrt{3}} + \frac{\pi}{48} + \frac{1}{64} \ln 3 - \frac{1}{24} \ln 2$$

$$- \frac{1}{24\sqrt{3}} \ln(2 + \sqrt{3}).$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)} \right.$$

$$\left. - \frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)} \right]$$

$$= -\frac{\pi}{18\sqrt{3}} + \frac{5\pi}{144},$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)} \right.$$

$$\left. - \frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)} \right]$$

$$= -\frac{37\pi}{2880\sqrt{3}} + \frac{11\pi}{1440} - \frac{3}{320} \ln 3 + \frac{1}{40} \ln 2$$

$$- \frac{1}{120\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)}$$

$$= \frac{\pi}{72\sqrt{3}} - \frac{23\pi}{2880} - \frac{1}{40} \ln 2 + \frac{11}{480\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-8)(12n-7)\cdots(12n-4)}$$

$$= -\frac{19\pi}{2880\sqrt{3}} + \frac{11\pi}{2880} + \frac{3}{320} \ln 3 + \frac{7}{360} \ln 2$$

$$- \frac{1}{32\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-8)(12n-7)(12n-6)\cdots(12n-3)}$$

$$= -\frac{19\pi}{2880\sqrt{3}} + \frac{11\pi}{2880} - \frac{3}{320} \ln 3 - \frac{7}{360} \ln 2$$

$$+ \frac{1}{32\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-7)(12n-6)(12n-5)\cdots(12n-2)}$$

$$= \frac{\pi}{72\sqrt{3}} - \frac{23\pi}{2880} + \frac{1}{40} \ln 2 - \frac{11}{480\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-6)(12n-5)(12n-4)\cdots(12n-1)}$$

$$= -\frac{37\pi}{2880\sqrt{3}} + \frac{11\pi}{1440} + \frac{3}{320} \ln 3 - \frac{1}{40} \ln 2$$

$$+ \frac{1}{120\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-4)(12n-3)\cdots 12n}$$

$$= \frac{23\pi}{2880\sqrt{3}} - \frac{11\pi}{1440} - \frac{3}{320} \ln 3 + \frac{7}{360} \ln 2$$

$$+ \frac{1}{120\sqrt{3}} \ln(2 + \sqrt{3}).$$

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-4)(6n-3)\cdots 6n} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)} \right. \\ & \quad \left. - \frac{1}{(12n-5)(12n-4)(12n-3)\cdots 12n} \right] \\ &= -\frac{\pi}{48\sqrt{3}} + \frac{11\pi}{720} + \frac{1}{180} \ln 2 - \frac{1}{60\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)} \\ &= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)} \right. \\ & \quad \left. - \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)} \right] \\ &= -\frac{77\pi}{17280\sqrt{3}} + \frac{\pi}{384} - \frac{1}{640} \ln 3 + \frac{1}{120} \ln 2 \\ & \quad - \frac{1}{192\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)} \\ &= \frac{59\pi}{17280\sqrt{3}} - \frac{17\pi}{8640} - \frac{1}{640} \ln 3 - \frac{1}{135} \ln 2 \\ & \quad + \frac{13}{1440\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-8)(12n-7)\cdots(12n-3)}$$



$$= \frac{1}{320} \ln 3 + \frac{7}{1080} \ln 2 - \frac{1}{96\sqrt{3}} \ln (2 + \sqrt{3}),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-8)(12n-7)(12n-6)\cdots(12n-2)} \\ &= -\frac{59\pi}{17280\sqrt{3}} + \frac{17\pi}{8640} - \frac{1}{640} \ln 3 - \frac{1}{135} \ln 2 \\ & \quad + \frac{13}{1440\sqrt{3}} \ln (2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-7)(12n-6)(12n-5)\cdots(12n-1)} \\ &= \frac{77\pi}{17280\sqrt{3}} - \frac{\pi}{384} + \frac{1}{120} \ln 2 - \frac{1}{640} \ln 3 \\ & \quad - \frac{1}{192\sqrt{3}} \ln (2 + \sqrt{3}); \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)} \\ &= \frac{1}{7} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)} \right. \\ & \quad \left. - \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)} \right] \\ &= -\frac{17\pi}{15120\sqrt{3}} + \frac{79\pi}{120960} + \frac{17}{7560} \ln 2 \\ & \quad - \frac{41}{20160\sqrt{3}} \ln (2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)} \\ &= \frac{59\pi}{120960\sqrt{3}} - \frac{17\pi}{60480} - \frac{3}{4480} \ln 3 \\ & \quad - \frac{1}{504} \ln 2 + \frac{1}{360\sqrt{3}} \ln (2 + \sqrt{3}), \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-8)(12n-7)\cdots(12n-2)}$$

$$= \frac{59\pi}{120960\sqrt{3}} - \frac{17\pi}{60480} + \frac{3}{4480} \ln 3$$

$$+ \frac{1}{504} \ln 2 - \frac{1}{306\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-8)(12n-7)(12n-6)\cdots(12n-1)}$$

$$= -\frac{17\pi}{15120\sqrt{3}} + \frac{79\pi}{120960} - \frac{17}{7560} \ln 2$$

$$+ \frac{41}{20160\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)}$$

$$= \frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)} \right.$$

$$\left. - \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)} \right]$$

$$= -\frac{13\pi}{64512\sqrt{3}} + \frac{113\pi}{967680} + \frac{1}{1890} \ln 2$$

$$+ \frac{3}{35840} \ln 3 - \frac{97}{161280\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)}$$

$$= -\frac{3}{17920} \ln 3 - \frac{1}{2016} \ln 2 + \frac{1}{1440\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-8)(12n-7)\cdots(12n-1)}$$

$$\begin{aligned}
&= -\frac{13\pi}{64512\sqrt{3}} - \frac{113\pi}{967680} + \frac{1}{1890} \ln 2 \\
&\quad + \frac{3}{35840} \ln 3 - \frac{97}{161280\sqrt{3}} \ln(2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)} \\
&= \frac{1}{9} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)} \right. \\
&\quad \left. - \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)} \right] \\
&= -\frac{13\pi}{580608\sqrt{3}} + \frac{113\pi}{8709120} + \frac{31}{272160} \ln 2 \\
&\quad + \frac{1}{35840} \ln 3 - \frac{209}{1451520\sqrt{3}} \ln(2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)} \\
&= -\frac{13\pi}{580608\sqrt{3}} + \frac{113\pi}{8709120} - \frac{31}{272160} \ln 2 \\
&\quad - \frac{1}{35840} \ln 3 + \frac{209}{1451520\sqrt{3}} \ln(2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-1)} \\
&= \frac{1}{10} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)} \right. \\
&\quad \left. - \frac{1}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)} \right] \\
&= \frac{31}{1360800} \ln 2 + \frac{1}{179200} \ln 3 \\
&\quad - \frac{209}{7257600\sqrt{3}} \ln(2 + \sqrt{3}).
\end{aligned}$$

$$\begin{aligned}
(五) \quad & \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-7)} \\
&= \frac{\pi}{96} + \frac{1}{32} \ln 3, \\
& \sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-7)(12n-5)} \\
&= \frac{\pi\sqrt{3}}{64} - \frac{5\pi}{192} - \frac{1}{64} \ln 3 + \frac{1}{32\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{1}{(12n-7)(12n-5)(12n-3)} \\
&= -\frac{\pi\sqrt{3}}{64} + \frac{5\pi}{192} - \frac{1}{64} \ln 3 + \frac{1}{32\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-3)(12n-1)} \\
&= -\frac{\pi}{96} + \frac{1}{32} \ln 3, \\
& \sum_{n=1}^{\infty} \frac{1}{(12n-3)(12n-1)(12n+1)} \\
&= \frac{\pi\sqrt{3}}{64} + \frac{5\pi}{192} - \frac{1}{64} \ln 3 - \frac{1}{32\sqrt{3}} \ln(2+\sqrt{3}) - \frac{1}{8}.
\end{aligned}$$

由此顺次求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-3)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-7)} \right. \\
&\quad \left. + \frac{1}{(12n-5)(12n-3)(12n-1)} \right] \\
&= \frac{1}{16} \ln 3,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-3)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-7)} \right. \\
&\quad \left. - \frac{1}{(12n-5)(12n-3)(12n-1)} \right] \\
&= \frac{\pi}{48},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)} \\
&= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-7)} \right. \\
&\quad \left. - \frac{1}{(12n-9)(12n-7)(12n-5)} \right] \\
&= -\frac{\pi}{128\sqrt{3}} + \frac{7\pi}{1152} + \frac{1}{128} \ln 3 \\
&\quad - \frac{1}{192\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-7)(12n-5)(12n-3)} \\
&= \frac{\pi}{64\sqrt{3}} - \frac{5\pi}{576},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(12n-7)(12n-5)(12n-3)(12n-1)} \\
&= -\frac{\pi}{128\sqrt{3}} + \frac{7\pi}{1152} - \frac{1}{128} \ln 3 \\
&\quad + \frac{1}{192\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$= -\frac{\pi}{128\sqrt{3}} - \frac{7\pi}{1152} + \frac{1}{128} \ln 3$$

$$+ \frac{1}{192\sqrt{3}} \ln (2 + \sqrt{3}) + \frac{1}{48},$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-3)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)} \right.$$

$$\left. + \frac{1}{(12n-5)(12n-3)(12n-1)(12n+1)} \right]$$

$$= -\frac{\pi}{64\sqrt{3}} + \frac{1}{64} \ln 3 + \frac{1}{48},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-3)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)} \right.$$

$$\left. - \frac{1}{(12n-5)(12n-3)(12n-1)(12n+1)} \right]$$

$$= \frac{7\pi}{576} - \frac{1}{96\sqrt{3}} \ln (2 + \sqrt{3}) - \frac{1}{48},$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)(12n-3)}$$

$$= \frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)} \right.$$

$$\left. - \frac{1}{(12n-9)(12n-7)(12n-5)(12n-3)} \right]$$

$$= -\frac{\pi\sqrt{3}}{1024} + \frac{17\pi}{9216} + \frac{1}{1024} \ln 3.$$

$$= \frac{1}{1536\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)} \\ &= \frac{\pi\sqrt{3}}{1024} - \frac{17\pi}{9216} + \frac{1}{1024} \ln 3 - \frac{1}{1536\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-7)(12n-5)(12n-3)(12n-1)(12n+1)} \\ &= \frac{7\pi}{4608} - \frac{1}{512} \ln 3 - \frac{1}{384}, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)} \\ &= \frac{1}{10} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-7)\cdots(12n-3)} \right. \\ & \quad \left. - \frac{1}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)} \right] \\ &= -\frac{\pi\sqrt{3}}{5120} + \frac{17\pi}{46080}, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)} \\ &= \frac{\pi\sqrt{3}}{10240} - \frac{31\pi}{92160} + \frac{3}{10240} \ln 3 \\ & \quad - \frac{1}{15360\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{3840}, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-7)\cdots(12n+1)} \\ &= \frac{1}{12} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)} \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)} \Big] \\
& = -\frac{\pi\sqrt{3}}{40960} + \frac{13\pi}{221184} - \frac{1}{40960} \ln 3 \\
& \quad + \frac{1}{184320\sqrt{3}} \ln(2+\sqrt{3}) - \frac{1}{46080}.
\end{aligned}$$

$$\begin{aligned}
(六) \quad & \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-8)(12n-5)} \\
& = \frac{\pi}{108\sqrt{3}} + \frac{1}{36} \ln 2, \\
& \sum_{n=1}^{\infty} \frac{1}{(12n-8)(12n-5)(12n-2)} \\
& = -\frac{\pi}{54\sqrt{3}} + \frac{\pi}{108} - \frac{1}{54} \ln 2 + \frac{1}{36\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-2)(12n+1)} \\
& = \frac{\pi}{36\sqrt{3}} + \frac{1}{108} \ln 2 - \frac{1}{18}.
\end{aligned}$$

由此顺次求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-2)(6n+1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-8)(12n-5)} \right. \\
& \quad \left. + \frac{1}{(12n-5)(12n-2)(12n+1)} \right] \\
& = \frac{\pi}{27\sqrt{3}} + \frac{1}{27} \ln 2 - \frac{1}{18}, \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-2)(6n+1)}
\end{aligned}$$



$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-8)(12n-5)} \right. \\
&\quad \left. - \frac{1}{(12n-5)(12n-2)(12n+1)} \right] \\
&= -\frac{\pi}{54\sqrt{3}} + \frac{1}{54} \ln 2 + \frac{1}{18}, \\
&\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-8)(12n-5)(12n-2)} \\
&= \frac{1}{9} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-8)(12n-5)} \right. \\
&\quad \left. - \frac{1}{(12n-8)(12n-5)(12n-2)} \right] \\
&= \frac{\pi}{324\sqrt{3}} - \frac{\pi}{972} + \frac{5}{972} \ln 2 \\
&\quad - \frac{1}{324\sqrt{3}} \ln(2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{1}{(12n-8)(12n-5)(12n-2)(12n+1)} \\
&= -\frac{5\pi}{972\sqrt{3}} + \frac{\pi}{972} - \frac{1}{324} \ln 2 \\
&\quad + \frac{1}{324\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{162}, \\
&\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-8)(12n-5)(12n-2)(12n+1)} \\
&= \frac{1}{12} \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-8)(12n-5)(12n-2)} \right. \\
&\quad \left. - \frac{1}{(12n-8)(12n-5)(12n-2)(12n+1)} \right] \\
&= \frac{\pi}{1458\sqrt{3}} - \frac{\pi}{5832} + \frac{1}{1458} \ln 2
\end{aligned}$$

$$= \frac{1}{1944\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{1944}.$$

例3 求和:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(8n-7)(8n-6)(8n-5)},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}.$$

解 于定理9令 $a=1$ ,  $b=2$ ,  $k=16$ , 则有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(16n-15)(16n-14)} &= \sum_{n=1}^{\infty} \left( \frac{1}{16n-15} - \frac{1}{16n-14} \right) \\ &= C(1, 16) - \frac{1}{2} C(1, 8) + \frac{1}{16} \ln 2. \end{aligned}$$

根据定理6与8, 有

$$C(1, 16) - \frac{1}{2} C(1, 8) = \frac{1}{2} A(1, 8).$$

$$\text{于是 } \sum_{n=1}^{\infty} \frac{1}{(16n-15)(16n-14)} = \frac{1}{2} A(1, 8) + \frac{1}{16} \ln 2.$$

仿此求得

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{(16n-14)(16n-13)} \\ &= \frac{\pi}{16} - \frac{1}{16} \ln 2 + \frac{1}{4\sqrt{2}} \ln(\sqrt{2} + 1) - \frac{1}{2} A(3, 8), \\ &\sum_{n=1}^{\infty} \frac{1}{(16n-13)(16n-12)} \\ &= \frac{\pi}{16\sqrt{2}} - \frac{\pi}{16} + \frac{1}{8} \ln 2 - \frac{1}{8\sqrt{2}} \ln(\sqrt{2} + 1) \end{aligned}$$

$$+ \frac{1}{2} A(3, 8),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-12)(16n-11)} \\ &= \frac{\pi}{16\sqrt{2}} - \frac{1}{8} \ln 2 + \frac{1}{8\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{2} A(5, 8) \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-7)(16n-6)} \\ &= \frac{\pi}{8\sqrt{2}} + \frac{1}{16} \ln 2 + \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{2} A(1, 8), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-6)(16n-5)} \\ &= -\frac{\pi}{8\sqrt{2}} + \frac{\pi}{16} - \frac{1}{16} \ln 2 + \frac{1}{2} A(3, 8), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-5)(16n-4)} \\ &= \frac{\pi}{16\sqrt{2}} + \frac{1}{8} \ln 2 - \frac{1}{8\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{2} A(3, 8), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-4)(16n-3)} \\ &= \frac{\pi}{16\sqrt{2}} - \frac{\pi}{16} - \frac{1}{8} \ln 2 + \frac{1}{8\sqrt{2}} \ln(\sqrt{2}+1) \\ & \quad + \frac{1}{2} A(5, 8). \end{aligned}$$

由此顺次求得

$$\sum_{n=1}^{\infty} \frac{1}{(16n-15)(16n-14)(16n-13)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(16n-15)(16n-14)} - \frac{1}{(16n-14)(16n-13)} \right]$$

$$= -\frac{\pi}{32} + \frac{1}{16} \ln 2 - \frac{1}{8\sqrt{2}} \ln(\sqrt{2} + 1) + \frac{1}{4} A(1,8) \\ + \frac{1}{4} A(3,8) = 0.1668812545\cdots,$$

$$\sum_{n=1}^{\infty} \frac{1}{(16n-14)(16n-13)(16n-12)}$$

$$= -\frac{\pi}{32\sqrt{2}} + \frac{\pi}{16} - \frac{3}{32} \ln 2 + \frac{3}{16\sqrt{2}} \ln(\sqrt{2} + 1) \\ - \frac{1}{2} A(3,8) = 0.0418525697\cdots,$$

$$\sum_{n=1}^{\infty} \frac{1}{(16n-13)(16n-12)(16n-11)}$$

$$= -\frac{\pi}{32} + \frac{1}{8} \ln 2 - \frac{1}{8\sqrt{2}} \ln(\sqrt{2} + 1) + \frac{1}{4} A(3,8) \\ + \frac{1}{4} A(5,8) = 0.0168290778\cdots,$$

$$\sum_{n=1}^{\infty} \frac{1}{(16n-7)(16n-6)(16n-5)}$$

$$= \frac{\pi}{8\sqrt{2}} - \frac{\pi}{32} + \frac{1}{16} \ln 2 + \frac{1}{8\sqrt{2}} \ln(\sqrt{2} + 1) \\ - \frac{1}{4} A(1,8) - \frac{1}{4} A(3,8) = 0.0010927858\cdots,$$

$$\sum_{n=1}^{\infty} \frac{1}{(16n-6)(16n-5)(16n-4)}$$

$$= -\frac{3\pi}{32\sqrt{2}} + \frac{\pi}{32} - \frac{3}{32} \ln 2 + \frac{1}{16\sqrt{2}} \ln(\sqrt{2} + 1)$$

$$+ \frac{1}{2} A(3, 8) = 0.00083277164\cdots,$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-5)(16n-4)(16n-3)} \\ &= \frac{\pi}{32} + \frac{1}{8} \ln 2 - \frac{1}{8\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{4} A(3, 8) \\ & \quad - \frac{1}{4} A(5, 8) = 0.00065140722\cdots. \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-15)(16n-14)(16n-13)(16n-12)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(16n-15)(16n-14)(16n-13)} \right. \\ & \quad \left. - \frac{1}{(16n-14)(16n-13)(16n-12)} \right] \\ &= \frac{\pi}{96\sqrt{2}} - \frac{\pi}{32} + \frac{5}{96} \ln 2 - \frac{5}{48\sqrt{2}} \ln(\sqrt{2}+1) \\ & \quad + \frac{1}{12} A(1, 8) + \frac{1}{4} A(3, 8) = 0.0416762282\cdots, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-14)(16n-13)(16n-12)(16n-11)} \\ &= -\frac{\pi}{96\sqrt{2}} + \frac{\pi}{32} - \frac{7}{96} \ln 2 + \frac{5}{48\sqrt{2}} \ln(\sqrt{2}+1) \\ & \quad - \frac{1}{4} A(3, 8) - \frac{1}{12} A(5, 8) = 0.0083411639\cdots, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-7)(16n-6)(16n-5)(16n-4)} \\ &= \frac{7\pi}{96\sqrt{2}} - \frac{\pi}{48} + \frac{5}{96} \ln 2 + \frac{1}{48\sqrt{2}} \ln(\sqrt{2}+1) \end{aligned}$$

$$-\frac{1}{12}A(1,8) - \frac{1}{4}A(3,8) = 0.000086671386\cdots,$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-6)(16n-5)(16n-4)(16n-3)} \\ &= -\frac{\pi}{32\sqrt{2}} - \frac{7}{96} \ln 2 + \frac{1}{16\sqrt{2}} \ln(\sqrt{2}+1) \\ &+ \frac{1}{4}A(3,8) + \frac{1}{12}A(5,8) = 0.000060454803\cdots. \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-15)(16n-14)(16n-13)\cdots(16n-11)} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{1}{(16n-15)(16n-14)(16n-13)(16n-12)} \right. \\ & \quad \left. - \frac{1}{(16n-14)(16n-13)(16n-12)(16n-11)} \right] \\ &= \frac{\pi}{192\sqrt{2}} - \frac{\pi}{64} + \frac{1}{32} \ln 2 - \frac{5}{96\sqrt{2}} \ln(\sqrt{2}+1) \\ &+ \frac{1}{48}A(1,8) + \frac{1}{8}A(3,8) + \frac{1}{48}A(5,8) \\ &= 0.0083337660\cdots, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(16n-7)(16n-6)(16n-5)(16n-4)(16n-3)} \\ &= \frac{5\pi}{192\sqrt{2}} - \frac{\pi}{192} + \frac{1}{32} \ln 2 - \frac{1}{96\sqrt{2}} \ln(\sqrt{2}+1) \\ & - \frac{1}{48}A(1,8) - \frac{1}{8}A(3,8) - \frac{1}{48}A(5,8) \\ &= 0.0000065541448\cdots. \end{aligned}$$

于是 
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(8n-7)(8n-6)(8n-5)}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(16n-15)(16n-14)(15n-13)} \right. \\
&\quad \left. - \frac{1}{(16n-7)(16n-6)(16n-5)} \right] \\
&= -\frac{\pi}{8\sqrt{2}} - \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{2} A(1,8) \\
&\quad + \frac{1}{2} A(3,8) = 0.1657884687\ldots,
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(16n-15)(16n-14)(16n-13)(16n-12)} \right. \\
&\quad \left. - \frac{1}{(16n-7)(16n-6)(16n-5)(16n-4)} \right] \\
&= -\frac{\pi}{16\sqrt{2}} - \frac{\pi}{96} - \frac{1}{8\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{6} A(1,8) \\
&\quad + \frac{1}{2} A(3,8) = 0.0415895569\ldots,
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(16n-15)(16n-14)(16n-13)\cdots(16n-11)} \right. \\
&\quad \left. - \frac{1}{(16n-7)(16n-6)(16n-5)(16n-4)(16n-3)} \right] \\
&= -\frac{\pi}{48\sqrt{2}} - \frac{\pi}{96} - \frac{1}{24\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{24} A(1,8) \\
&\quad + \frac{1}{4} A(3,8) + \frac{1}{24} A(5,8) = 0.0083272119\ldots.
\end{aligned}$$

例4 求和:

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} + \frac{1}{12n-7} - \frac{1}{12n-5} - \frac{1}{12n-1} \right),$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{1}{12n-7} - \frac{1}{12n-5} + \frac{1}{12n-1} \right).$$

解 于定理 9 令  $a=1$ ,  $b=7$ ,  $k=12$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{1}{12n-5} \right) = C(1,12) - C(7,12)$$

$$+ \frac{1}{12} \ln 7.$$

根据定理 6, 有

$$C(1,12) - C(7,12) = \frac{\pi}{6} - \frac{1}{12} \ln 7$$

$$+ \frac{1}{2\sqrt{3}} \ln (2 + \sqrt{3}).$$

于是 
$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{1}{12n-5} \right) = \frac{\pi}{6} + \frac{1}{2\sqrt{3}} \ln (2 + \sqrt{3}).$$

仿此求得

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-7} - \frac{1}{12n-1} \right) = \frac{\pi}{6} - \frac{1}{2\sqrt{3}} \ln (2 + \sqrt{3}).$$

于是 
$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} + \frac{1}{12n-7} - \frac{1}{12n-5} - \frac{1}{12n-1} \right) = \frac{\pi}{3},$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{1}{12n-7} - \frac{1}{12n-5} + \frac{1}{12n-1} \right)$$

$$= \frac{1}{\sqrt{3}} \ln (2 + \sqrt{3}).$$

仿此求得



$$\sum_{n=1}^{\infty} \left( \frac{1}{4n-3} + \frac{1}{4n-2} - \frac{1}{4n-1} - \frac{1}{4n} \right) = \frac{\pi}{4} + \frac{1}{2} \ln 2,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{4n-3} - \frac{1}{4n-2} - \frac{1}{4n-1} + \frac{1}{4n} \right) = \frac{\pi}{4} - \frac{1}{2} \ln 2,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{8n-7} + \frac{1}{8n-5} - \frac{1}{8n-3} - \frac{1}{8n-1} \right) = \frac{\pi}{2\sqrt{2}},$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} - \frac{1}{8n-5} - \frac{1}{8n-3} + \frac{1}{8n-1} \right) \\ = \frac{1}{\sqrt{2}} \ln (\sqrt{2} + 1); \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{1}{6n-5} + \frac{1}{6n-4} + \frac{1}{6n-3} - \frac{1}{6n-2} - \frac{1}{6n-1} - \frac{1}{6n} \right) \\ = \frac{2\pi}{3\sqrt{3}} + \frac{1}{3} \ln 2; \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{1}{12n-11} + \frac{1}{12n-9} + \frac{1}{12n-7} - \frac{1}{12n-5} - \frac{1}{12n-3} \right. \\ \left. - \frac{1}{12n-1} \right) = \frac{5\pi}{12}, \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} + \frac{1}{8n-6} + \frac{1}{8n-5} + \frac{1}{8n-4} - \frac{1}{8n-3} \right. \\ \left. - \frac{1}{8n-2} - \frac{1}{8n-1} - \frac{1}{8n} \right) \\ = \frac{\pi}{2\sqrt{2}} + \frac{\pi}{8} + \frac{1}{4} \ln 2, \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} + \frac{1}{8n-6} - \frac{1}{8n-5} - \frac{1}{8n-4} - \frac{1}{8n-3} \right. \\ \left. - \frac{1}{8n-2} + \frac{1}{8n-1} + \frac{1}{8n} \right) \\ = \frac{\pi}{8} - \frac{1}{4} \ln 2 + \frac{1}{\sqrt{2}} \ln (\sqrt{2} + 1); \end{aligned}$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} + \frac{1}{12n-10} + \cdots + \frac{1}{12n-7} - \frac{1}{12n-6} \right. \\ \left. - \frac{1}{12n-5} - \cdots - \frac{1}{12n-1} \right)$$

$$= \frac{5\pi}{12} + \frac{\pi}{3\sqrt{3}},$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} + \frac{1}{12n-10} + \cdots + \frac{1}{12n-6} - \frac{1}{12n-5} \right. \\ \left. - \frac{1}{12n-4} - \cdots - \frac{1}{12n} \right)$$

$$= \frac{5\pi}{12} + \frac{\pi}{3\sqrt{3}} + \frac{1}{6} \ln 2,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} + \frac{1}{12n-10} + \frac{1}{12n-9} - \frac{1}{12n-8} - \frac{1}{12n-7} \right. \\ \left. - \cdots - \frac{1}{12n-3} + \frac{1}{12n-2} + \frac{1}{12n-1} + \frac{1}{12n} \right)$$

$$= \frac{\pi}{12} + \frac{1}{6} \ln 2 + \frac{1}{\sqrt{3}} \ln(2 + \sqrt{3});$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{16n-15} - \frac{1}{16n-13} + \frac{1}{16n-7} - \frac{1}{16n-5} \right)$$

$$= \frac{\pi}{8} + \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{16n-11} - \frac{1}{16n-9} + \frac{1}{16n-3} - \frac{1}{16n-1} \right)$$

$$= \frac{\pi}{8} - \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{16n-15} - \frac{1}{16n-9} + \frac{1}{16n-7} - \frac{1}{16n-1} \right)$$

$$= \frac{\pi}{4\sqrt{2}} + \frac{\pi}{8},$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{16n-13} - \frac{1}{16n-11} + \frac{1}{16n-5} - \frac{1}{16n-3} \right) \\ = \frac{\pi}{4\sqrt{2}} - \frac{\pi}{8}.$$

例5 求和:

$$\sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{1}{24n-19} + \frac{1}{24n-11} - \frac{1}{24n-7} \right).$$

解 于定理9 令  $a=1$ ,  $b=5$ ,  $k=24$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{1}{24n-19} \right) = C(1,24) - C(5,24) \\ + \frac{1}{24} \ln 5 \\ = \left[ C(1,24) - \frac{1}{2} C(1,12) \right] - \left[ C(5,24) - \frac{1}{2} C(5,12) \right] \\ + \frac{1}{2} C(1,12) - \frac{1}{2} C(5,12) + \frac{1}{24} \ln 5.$$

根据定理5, 有

$$C(1,24) - \frac{1}{2} C(1,12) = \frac{1}{2} A(1,12),$$

$$C(5,24) - \frac{1}{2} C(5,12) = \frac{1}{2} A(5,12).$$

$$\text{于是 } \sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{1}{24n-19} \right) \\ = \frac{1}{2} A(1,12) - \frac{1}{2} A(5,12) + \frac{1}{2} C(1,12) - \frac{1}{2} C(5,12) \\ + \frac{1}{24} \ln 5.$$

于定理9 令  $a=13$ ,  $b=17$ ,  $k=24$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left( \frac{1}{24n-11} - \frac{1}{24n-7} \right) \\
&= C(13, 24) - C(17, 24) + \frac{1}{24} \ln \frac{17}{13} \\
&= \left[ C(13, 24) - \frac{1}{2} C(13, 12) \right] \\
&\quad - \left[ C(17, 24) - \frac{1}{2} C(17, 12) \right] + \frac{1}{2} C(13, 12) \\
&\quad - \frac{1}{2} C(17, 12) + \frac{1}{24} \ln \frac{17}{13} \\
&= \frac{1}{2} A(13, 12) - \frac{1}{2} A(17, 12) + \frac{1}{2} C(13, 12) \\
&\quad - \frac{1}{2} C(17, 12) + \frac{1}{24} \ln \frac{17}{13}.
\end{aligned}$$

$$A(13, 12) = -A(1, 12) + 1,$$

$$A(17, 12) = -A(5, 12) + \frac{1}{5},$$

$$\begin{aligned}
& \frac{1}{2} A(13, 12) - \frac{1}{2} A(17, 12) \\
&= -\frac{1}{2} A(1, 12) + \frac{1}{2} A(5, 12) + \frac{2}{5},
\end{aligned}$$

于是 
$$\begin{aligned}
& \sum_{n=1}^{\infty} \left( \frac{1}{24n-11} - \frac{1}{24n-7} \right) \\
&= -\frac{1}{2} A(1, 12) + \frac{1}{2} A(5, 12) + \frac{2}{5} + \frac{1}{2} C(13, 12) \\
&\quad - \frac{1}{2} C(17, 12) + \frac{1}{24} \ln \frac{17}{13}.
\end{aligned}$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{1}{24n-19} + \frac{1}{24n-11} - \frac{1}{24n-7} \right)$$

$$= \frac{1}{2} C(1, 12) - \frac{1}{2} C(5, 12) + \frac{1}{2} C(13, 12) \\ - \frac{1}{2} C(17, 12) + \frac{1}{24} \ln \frac{5 \cdot 17}{13} + \frac{2}{5}.$$

于命题 2 令  $a = 1, b = 1, k = 12$ , 则有

$$C(13, 12) = C(1, 12) + \frac{1}{12} \ln 13 - 1,$$

令  $a = 5, b = 1, k = 12$ , 则有

$$C(17, 12) = C(5, 12) + \frac{1}{12} \ln \frac{17}{5} - \frac{1}{5}.$$

于是  $\frac{1}{2} C(13, 12) - \frac{1}{2} C(17, 12)$

$$= \frac{1}{2} C(1, 12) - \frac{1}{2} C(5, 12) + \frac{1}{24} \ln \frac{5 \cdot 13}{17} - \frac{2}{5}.$$

$$\text{从而 } \sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{1}{24n-19} + \frac{1}{24n-11} - \frac{1}{24n-7} \right) \\ = C(1, 12) - C(5, 12) + \frac{1}{12} \ln 5.$$

根据定理 6, 有

$$C(1, 12) - C(5, 12) = \frac{\pi}{4\sqrt{3}} - \frac{1}{12} \ln 5 \\ + \frac{1}{2\sqrt{3}} \ln (2 + \sqrt{3}).$$

$$\text{所以 } \sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{1}{24n-19} + \frac{1}{24n-11} - \frac{1}{24n-7} \right) \\ = \frac{\pi}{4\sqrt{3}} + \frac{1}{2\sqrt{3}} \ln (2 + \sqrt{3}).$$

仿此求得

$$\sum_{n=1}^{\infty} \left( \frac{1}{24n-17} - \frac{1}{24n-13} + \frac{1}{24n-5} - \frac{1}{24n-1} \right)$$

$$= \frac{\pi}{4\sqrt{3}} - \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{1}{24n-13} + \frac{1}{24n-11} - \frac{1}{24n-1} \right)$$

$$= \frac{\pi}{4\sqrt{3}} + \frac{\pi}{6},$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{24n-19} - \frac{1}{24n-17} + \frac{1}{24n-7} - \frac{1}{24n-5} \right)$$

$$= -\frac{\pi}{4\sqrt{3}} + \frac{\pi}{6}.$$

$$2. \sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{B-A}{k(n-1)+d} \right]$$

**定理10** 设  $A, B, b, d$  都是正整数, 则

$$\sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{B-A}{k(n-1)+d} \right]$$

$$= AC(a, k) - BC(b, k) + (B-A)C(d, k)$$

$$+ \frac{1}{k} \ln \frac{b^B}{a^A d^{B-A}}.$$

**证明** 根据定理2, 有

$$\sum_{i=1}^n \left[ \frac{A}{k(i-1)+a} - \frac{B}{k(i-1)+b} + \frac{B-A}{k(i-1)+d} \right]$$

$$= AC(a, k) - BC(b, k) + (B-A)C(d, k)$$

$$+ \frac{1}{k} \ln \frac{(nk-k+a)^A (nk-k+d)^{B-A}}{(nk-k+b)^B}$$

$$+ \frac{1}{k} \ln \frac{b^B}{a^A d^{B-A}} + A\varepsilon_n(a, k) - B\varepsilon_n(b, k) \\ + (B-A)\varepsilon_n(d, k).$$

令  $n \rightarrow \infty$  并取极限, 定理即得证.

**例1** 求和  $\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-4)(8n-1)}.$

**解** 用分项分式方法求得

$$\frac{1}{(8n-7)(8n-4)(8n-1)} \\ = \frac{1}{18} \left( \frac{1}{8n-7} - \frac{2}{8n-4} + \frac{1}{8n-1} \right).$$

于定理10令  $A=1, B=2, a=1, b=4, d=7, k=8$ , 则有

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-4)(8n-1)} \\ = \frac{1}{18} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} - \frac{2}{8n-4} + \frac{1}{8n-1} \right) \\ = \frac{1}{18} C(1, 8) - \frac{1}{36} C(1, 2) + \frac{1}{18} C(7, 8) + \frac{1}{144} \ln \frac{2^4}{7}.$$

根据定理6, 有

$$\frac{1}{18} C(1, 8) - \frac{1}{72} C(1, 2) \\ = \frac{\pi}{144\sqrt{2}} + \frac{\pi}{288} + \frac{1}{72\sqrt{2}} \ln(\sqrt{2}+1), \\ \frac{1}{18} C(7, 8) - \frac{1}{72} C(1, 2) \\ = -\frac{\pi}{144\sqrt{2}} - \frac{\pi}{288} + \frac{1}{144} \ln 7$$

$$+ \frac{1}{72\sqrt{2}} \ln(\sqrt{2} + 1).$$

于是 
$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-4)(8n-1)}$$
  

$$= \frac{1}{36} \ln 2 + \frac{1}{36\sqrt{2}} \ln(\sqrt{2} + 1).$$

例2 求和 
$$\sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(6n-3)\cdots(6n-1)}$$

解 用分项分式方法求得

$$\frac{n}{(6n-5)(6n-4)(6n-3)}$$

$$= \frac{1}{12} \left( \frac{5}{6n-5} - \frac{8}{6n-4} + \frac{3}{6n-3} \right).$$

于定理 10 令  $A=5$ ,  $B=8$ ,  $a=1$ ,  $b=2$ ,  $d=3$ ,  $k=6$ , 则有

$$\sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(6n-3)}$$

$$= \frac{1}{12} \sum_{n=1}^{\infty} \left( \frac{5}{6n-5} - \frac{8}{6n-4} + \frac{3}{6n-3} \right)$$

$$= \frac{5}{12} C(1,6) - \frac{1}{3} C(1,3) + \frac{1}{12} C(1,2) + \frac{1}{72} \ln \frac{2^8}{3^3}.$$

根据定理 6, 有

$$\frac{5}{12} C(1,6) - \frac{5}{24} C(1,3) = \frac{5\pi}{72\sqrt{3}} + \frac{5}{72} \ln 2,$$

$$\frac{1}{12} C(1,2) - \frac{1}{8} C(1,3) = -\frac{\pi}{48\sqrt{3}} + \frac{1}{48} \ln \frac{2^2}{3}.$$

于是 
$$\sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(6n-3)}$$



$$= \frac{7\pi}{144\sqrt{3}} + \frac{2}{9} \ln 2 - \frac{1}{16} \ln 3.$$

仿此求得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(6n-4)(6n-3)(6n-2)} \\ &= \frac{1}{6} \sum_{n=1}^{\infty} \left( \frac{2}{6n-4} - \frac{3}{6n-3} + \frac{1}{6n-2} \right) \\ &= \frac{\pi}{72\sqrt{3}} + \frac{1}{8} \ln 3 - \frac{1}{6} \ln 2, \\ & \sum_{n=1}^{\infty} \frac{n}{(6n-3)(6n-2)(6n-1)} \\ &= \frac{1}{12} \sum_{n=1}^{\infty} \left( \frac{3}{6n-3} - \frac{4}{6n-2} + \frac{1}{6n-1} \right) \\ &= \frac{\pi}{144\sqrt{3}} + \frac{1}{9} \ln 2 - \frac{1}{16} \ln 3. \end{aligned}$$

由此顺次求得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-5)(6n-4)(6n-3)} \right. \\ & \quad \left. - \frac{n}{(6n-4)(6n-3)(6n-2)} \right] \\ &= \frac{5\pi}{432\sqrt{3}} + \frac{7}{54} \ln 2 - \frac{1}{16} \ln 3, \\ & \sum_{n=1}^{\infty} \frac{n}{(6n-4)(6n-3)(6n-2)(6n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-4)(6n-3)(6n-2)} \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{n}{(6n-3)(6n-2)(6n-1)} \Big] \\
& = \frac{\pi}{432\sqrt{3}} + \frac{1}{16} \ln 3 - \frac{5}{54} \ln 2; \\
& \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)} \\
& = \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)} \right. \\
& \quad \left. - \frac{n}{(6n-4)(6n-3)(6n-2)(6n-1)} \right] \\
& = \frac{\pi}{432\sqrt{3}} + \frac{1}{18} \ln 2 - \frac{1}{32} \ln 3.
\end{aligned}$$

仿此求得

$$\begin{aligned}
(-) \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)} \\
& = \frac{1}{16} \sum_{n=1}^{\infty} \left( \frac{7}{8n-7} - \frac{12}{8n-6} + \frac{5}{8n-5} \right) \\
& = \frac{3\pi}{32\sqrt{2}} - \frac{5\pi}{128} + \frac{3}{32} \ln 2 + \frac{1}{32\sqrt{2}} \ln(\sqrt{2}+1); \\
& \sum_{n=1}^{\infty} \frac{n}{(8n-6)(8n-5)(8n-4)} \\
& = \frac{1}{16} \sum_{n=1}^{\infty} \left( \frac{6}{8n-6} - \frac{10}{8n-5} + \frac{4}{8n-4} \right) \\
& = -\frac{5\pi}{64\sqrt{2}} + \frac{\pi}{16} - \frac{7}{64} \ln 2 + \frac{5}{32\sqrt{2}} \ln(\sqrt{2}+1); \\
& \sum_{n=1}^{\infty} \frac{n}{(8n-5)(8n-4)(8n-3)} \\
& = \frac{1}{16} \sum_{n=1}^{\infty} \left( \frac{5}{8n-5} - \frac{8}{8n-4} + \frac{3}{8n-3} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{64\sqrt{2}} - \frac{\pi}{128} + \frac{1}{8} \ln 2 - \frac{1}{8\sqrt{2}} \ln(\sqrt{2} + 1); \\
&\sum_{n=1}^{\infty} \frac{n}{(8n-4)(8n-3)(8n-2)} \\
&= \frac{1}{16} \sum_{n=1}^{\infty} \left( \frac{4}{8n-4} - \frac{6}{8n-3} + \frac{2}{8n-2} \right) \\
&= \frac{3\pi}{64\sqrt{2}} - \frac{\pi}{32} - \frac{5}{64} \ln 2 + \frac{3}{32\sqrt{2}} \ln(\sqrt{2} + 1); \\
&\sum_{n=1}^{\infty} \frac{n}{(8n-3)(8n-2)(8n-1)} \\
&= \frac{1}{16} \sum_{n=1}^{\infty} \left( \frac{3}{8n-3} - \frac{4}{8n-2} + \frac{1}{8n-1} \right) \\
&= -\frac{\pi}{32\sqrt{2}} + \frac{3\pi}{128} + \frac{1}{32} \ln 2 - \frac{1}{32\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

由此顺次求得

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(4n-3)(4n-2)(4n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-7)(8n-6)(8n-5)} \right. \\
&\quad \left. - \frac{2n}{(8n-3)(8n-2)(8n-1)} \right] \\
&= 2 \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)} \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)} \\
&\quad - 2 \sum_{n=1}^{\infty} \frac{n}{(8n-3)(8n-2)(8n-1)} \\
&= \frac{\pi}{8\sqrt{2}} - \frac{\pi}{16} + \frac{1}{8\sqrt{2}} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-2)(4n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-7)(8n-6)(8n-5)} \right. \\
&\quad \left. + \frac{2n}{(8n-3)(8n-2)(8n-1)} \right] \\
&= \frac{\pi}{32} + \frac{1}{8} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(8n-7)(8n-6)(8n-5)} \right. \\
&\quad \left. - \frac{n}{(8n-6)(8n-5)(8n-4)} \right] \\
&= \frac{11\pi}{192\sqrt{2}} - \frac{13\pi}{384} + \frac{13}{192} \ln 2 - \frac{1}{24\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-6)(8n-5)(8n-4)(8n-3)} \\
&= -\frac{\pi}{32\sqrt{2}} + \frac{3\pi}{128} - \frac{5}{64} \ln 2 + \frac{3}{32\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-5)(8n-4)(8n-3)(8n-2)} \\
&= -\frac{\pi}{96\sqrt{2}} + \frac{\pi}{128} + \frac{13}{192} \ln 2 - \frac{7}{96\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-4)(8n-3)(8n-2)(8n-1)} \\
&= \frac{5\pi}{192\sqrt{2}} - \frac{7\pi}{384} - \frac{7}{192} \ln 2 + \frac{1}{24\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)} \right. \\ \left. - \frac{n}{(8n-6)(8n-5)(8n-4)(8n-3)} \right]$$

$$= \frac{17\pi}{768\sqrt{2}} - \frac{11\pi}{768} + \frac{7}{192} \ln 2 \\ - \frac{13}{384\sqrt{2}} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{n}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)} \\ = -\frac{\pi}{192\sqrt{2}} + \frac{\pi}{256} - \frac{7}{192} \ln 2 \\ + \frac{1}{24\sqrt{2}} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{n}{(8n-5)(8n-4)(8n-3)(8n-2)(8n-1)} \\ = -\frac{7\pi}{768\sqrt{2}} + \frac{5\pi}{768} + \frac{5}{192} \ln 2 \\ - \frac{11}{384\sqrt{2}} \ln(\sqrt{2}+1);$$

$$\sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)} \\ = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \right. \\ \left. - \frac{n}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)} \right] \\ = \frac{7\pi}{1280\sqrt{2}} - \frac{7\pi}{1920} + \frac{7}{480} \ln 2 - \frac{29}{1920\sqrt{2}} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{n}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}$$

$$= \frac{\pi}{1280\sqrt{2}} - \frac{\pi}{1920} - \frac{1}{80} \ln 2 + \frac{9}{640\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}$$

$$= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{n}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)} \right.$$

$$\left. - \frac{n}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)} \right]$$

$$= \frac{\pi}{1280\sqrt{2}} - \frac{\pi}{1920} + \frac{13}{2880} \ln 2$$

$$- \frac{7}{1440\sqrt{2}} \ln(\sqrt{2} + 1).$$

$$(二) \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-5)(8n-3)}$$

$$= \frac{1}{64} \sum_{n=1}^{\infty} \left( \frac{7}{8n-7} - \frac{10}{8n-5} + \frac{3}{8n-3} \right)$$

$$= -\frac{3\pi}{256\sqrt{2}} + \frac{5\pi}{256} + \frac{7}{128\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n}{(8n-5)(8n-3)(8n-1)}$$

$$= \frac{1}{64} \sum_{n=1}^{\infty} \left( \frac{5}{8n-5} - \frac{6}{8n-3} + \frac{1}{8n-1} \right)$$

$$= \frac{5\pi}{256\sqrt{2}} - \frac{3\pi}{256} + \frac{1}{128\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n}{(8n-3)(8n-1)(8n+1)}$$

$$= \frac{1}{64} \sum_{n=1}^{\infty} \left( \frac{3}{8n-3} - \frac{2}{8n-1} - \frac{1}{8n+1} \right)$$

$$= -\frac{\pi}{256\sqrt{2}} + \frac{\pi}{256} - \frac{3}{128\sqrt{2}} \ln(\sqrt{2} + 1) + \frac{1}{64}.$$

由此顺次求得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(4n-3)(4n-1)(4n+1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-7)(8n-5)(8n-3)} - \frac{2n}{(8n-3)(8n-1)(8n+1)} \right]$$

$$= 2 \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-5)(8n-3)}$$

$$- \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(8n-3)}$$

$$- 2 \sum_{n=1}^{\infty} \frac{n}{(8n-3)(8n-1)(8n+1)}$$

$$= \frac{\pi}{64\sqrt{2}} + \frac{3}{32\sqrt{2}} \ln(\sqrt{2} + 1) - \frac{1}{32},$$

$$\sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-1)(4n+1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-7)(8n-5)(8n-3)} \right.$$

$$\left. + \frac{2n}{(8n-5)(8n-3)(8n-1)} \right]$$

$$= \frac{\pi}{64} + \frac{1}{32},$$

$$\sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-5)(8n-3)(8n-1)}$$

$$= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{n}{(8n-7)(8n-5)(8n-3)} \right.$$

$$\begin{aligned}
& - \frac{n}{(8n-5)(8n-3)(8n-1)} \Big] \\
& = -\frac{\pi}{192\sqrt{2}} + \frac{\pi}{192} + \frac{1}{128\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{n}{(8n-5)(8n-3)(8n-1)(8n+1)} \\
& = \frac{\pi}{256\sqrt{2}} - \frac{\pi}{384} + \frac{1}{192\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{384}, \\
& \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-5)(8n-3)(8n-1)(8n+1)} \\
& = \frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{n}{(8n-7)(8n-5)(8n-3)(8n-1)} \right. \\
& \quad \left. - \frac{n}{(8n-5)(8n-3)(8n-1)(8n+1)} \right] \\
& = -\frac{7\pi}{6144\sqrt{2}} + \frac{\pi}{1024} + \frac{1}{3072\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{3072}.
\end{aligned}$$

$$\begin{aligned}
(\text{三}) \quad & \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)} \\
& = \frac{1}{24} \sum_{n=1}^{\infty} \left( \frac{11}{12n-11} - \frac{20}{12n-10} + \frac{9}{12n-9} \right) \\
& = -\frac{\pi\sqrt{3}}{64} + \frac{31\pi}{576} + \frac{5}{72} \ln 2 - \frac{3}{64} \ln 3 \\
& \quad + \frac{11}{96\sqrt{3}} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-10)(12n-9)(12n-8)} \\
& = \frac{1}{12} \sum_{n=1}^{\infty} \left( \frac{5}{12n-10} - \frac{9}{12n-9} + \frac{4}{12n-8} \right)
\end{aligned}$$



$$= \frac{19\pi}{288\sqrt{3}} - \frac{\pi}{32} + \frac{3}{32} \ln 3 - \frac{17}{144} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-8)(12n-7)}$$

$$= \frac{1}{24} \sum_{n=1}^{\infty} \left( \frac{9}{12n-9} - \frac{16}{12n-8} + \frac{7}{12n-7} \right)$$

$$= -\frac{37\pi}{576\sqrt{3}} + \frac{23\pi}{576} + \frac{1}{6} \ln 2 - \frac{3}{64} \ln 3$$

$$- \frac{7}{96\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-8)(12n-7)(12n-6)}$$

$$= \frac{1}{12} \sum_{n=1}^{\infty} \left( \frac{4}{12n-8} - \frac{7}{12n-7} + \frac{3}{12n-6} \right)$$

$$= \frac{25\pi}{288\sqrt{3}} - \frac{7\pi}{144} - \frac{5}{48} \ln 2 - \frac{1}{32} \ln 3$$

$$+ \frac{7}{48\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-7)(12n-6)(12n-5)}$$

$$= \frac{1}{24} \sum_{n=1}^{\infty} \left( \frac{7}{12n-7} - \frac{12}{12n-6} + \frac{5}{12n-5} \right)$$

$$= -\frac{\pi}{96\sqrt{3}} + \frac{\pi}{144} + \frac{1}{16} \ln 3 + \frac{1}{24} \ln 2$$

$$- \frac{1}{8\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-6)(12n-5)(12n-4)}$$

$$= \frac{1}{12} \sum_{n=1}^{\infty} \left( \frac{3}{12n-6} - \frac{5}{12n-5} + \frac{2}{12n-4} \right)$$

$$= -\frac{17\pi}{288\sqrt{3}} + \frac{5\pi}{144} - \frac{1}{16} \ln 2 - \frac{1}{32} \ln 3 \\ + \frac{5}{48\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-3)} \\ = \frac{1}{24} \sum_{n=1}^{\infty} \left( \frac{5}{12n-5} - \frac{8}{12n-4} + \frac{3}{12n-3} \right) \\ = \frac{23\pi}{576\sqrt{3}} - \frac{13\pi}{576} + \frac{1}{12} \ln 2 - \frac{1}{64} \ln 3 \\ - \frac{5}{96\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-4)(12n-3)(12n-2)} \\ = \frac{1}{12} \sum_{n=1}^{\infty} \left( \frac{2}{12n-4} - \frac{3}{12n-3} + \frac{1}{12n-2} \right) \\ = -\frac{5\pi}{288\sqrt{3}} + \frac{\pi}{96} + \frac{1}{32} \ln 3 - \frac{7}{144} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-3)(12n-2)(12n-1)} \\ = \frac{1}{24} \sum_{n=1}^{\infty} \left( \frac{3}{12n-3} - \frac{4}{12n-2} + \frac{1}{12n-1} \right) \\ = \frac{\pi}{64\sqrt{3}} - \frac{5\pi}{576} + \frac{1}{72} \ln 2 - \frac{1}{64} \ln 3 \\ + \frac{1}{96\sqrt{3}} \ln(2 + \sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-4)(6n-3)}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)(12n-9)} \right. \\
&\quad \left. - \frac{2n}{(12n-5)(12n-4)(12n-3)} \right] \\
&= 2 \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)} \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)} \\
&\quad - 2 \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-3)} \\
&= -\frac{\pi}{9\sqrt{3}} + \frac{13\pi}{144} - \frac{1}{9} \ln 2 + \frac{5}{24\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-10)(12n-9)} \right. \\
&\quad \left. - \frac{n}{(12n-10)(12n-9)(12n-8)} \right] \\
&= -\frac{65\pi}{1728\sqrt{3}} + \frac{49\pi}{1728} + \frac{1}{16} \ln 2 - \frac{3}{64} \ln 3 \\
&\quad + \frac{11}{288\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)} \\
&= \frac{25\pi}{576\sqrt{3}} - \frac{41\pi}{1728} + \frac{3}{64} \ln 3 - \frac{41}{432} \ln 2 \\
&\quad + \frac{7}{288\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$= -\frac{29\pi}{576\sqrt{3}} + \frac{17\pi}{576} + \frac{13}{144} \ln 2 - \frac{1}{192} \ln 3 \\ - \frac{7}{96\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-8)(12n-7)(12n-6)(12n-5)} \\ = \frac{7\pi}{216\sqrt{3}} - \frac{\pi}{54} - \frac{7}{144} \ln 2 - \frac{1}{32} \ln 3 \\ + \frac{13}{144\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-7)(12n-6)(12n-5)(12n-4)} \\ = \frac{7\pi}{432\sqrt{3}} - \frac{\pi}{108} + \frac{1}{32} \ln 3 + \frac{5}{144} \ln 2 \\ - \frac{11}{144\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-6)(12n-5)(12n-4)(12n-3)} \\ = -\frac{19\pi}{576\sqrt{3}} + \frac{11\pi}{576} - \frac{7}{144} \ln 2 - \frac{1}{192} \ln 3 \\ + \frac{5}{96\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)} \\ = \frac{11\pi}{576\sqrt{3}} - \frac{19\pi}{1728} + \frac{19}{432} \ln 2 - \frac{1}{64} \ln 3 \\ - \frac{5}{288\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-4)(12n-3)(12n-2)(12n-1)}$$

$$= -\frac{19\pi}{1728\sqrt{3}} + \frac{11\pi}{1728} + \frac{1}{64} \ln 3 - \frac{1}{48} \ln 2$$

$$- \frac{1}{288\sqrt{3}} \ln(2 + \sqrt{3}).$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-8)} \right.$$

$$\left. - \frac{2n}{(12n-5)(12n-4)(12n-3)(12n-2)} \right]$$

$$= 2 \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$- \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)}$$

$$- 2 \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$= -\frac{7\pi}{108\sqrt{3}} + \frac{19\pi}{432} - \frac{5}{108} \ln 2$$

$$+ \frac{5}{72\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)} \right.$$

$$\left. - \frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)} \right]$$

$$= -\frac{35\pi}{1728\sqrt{3}} + \frac{5\pi}{384} + \frac{17}{432} \ln 2 - \frac{3}{128} \ln 3$$

$$\begin{aligned}
& + \frac{1}{288\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)} \\
& = \frac{\pi\sqrt{3}}{128} - \frac{23\pi}{1728} + \frac{5}{384} \ln 3 - \frac{5}{108} \ln 2 \\
& \quad + \frac{7}{288\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-8)(12n-7)(12n-6)(12n-5)} \\
& = -\frac{143\pi}{6912\sqrt{3}} + \frac{83\pi}{6912} + \frac{5}{144} \ln 2 + \frac{5}{768} \ln 3 \\
& \quad - \frac{47}{1152\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-8)(12n-7)(12n-6)(12n-5)(12n-4)} \\
& = \frac{7\pi}{1728\sqrt{3}} - \frac{\pi}{432} - \frac{1}{48} \ln 2 - \frac{1}{64} \ln 3 \\
& \quad + \frac{1}{24\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-7)(12n-6)(12n-5)(12n-4)(12n-3)} \\
& = \frac{85\pi}{6912\sqrt{3}} - \frac{49\pi}{6912} + \frac{7}{768} \ln 3 + \frac{1}{48} \ln 2 \\
& \quad - \frac{37}{1152\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-6)(12n-5)(12n-4)(12n-3)(12n-2)} \\
& = -\frac{5\pi}{384\sqrt{3}} + \frac{13\pi}{1728} - \frac{5}{216} \ln 2 + \frac{1}{384} \ln 3
\end{aligned}$$

$$+ \frac{5}{288\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)} \\ &= \frac{13\pi}{1728\sqrt{3}} - \frac{5\pi}{1152} + \frac{7}{432} \ln 2 - \frac{1}{128} \ln 3 \\ & \quad - \frac{1}{288\sqrt{3}} \ln(2 + \sqrt{3}). \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)} \right. \\ & \quad \left. - \frac{2n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)} \right] \\ &= 2 \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)} \\ & \quad - \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)} \\ & \quad - 2 \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-3)\cdots(12n-1)} \\ &= -\frac{\pi}{36\sqrt{3}} + \frac{5\pi}{288} - \frac{1}{108} \ln 2 + \frac{1}{72\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)} \\ &= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)} \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)} \Big] \\
& = - \frac{151\pi}{17280\sqrt{3}} + \frac{91\pi}{17280} + \frac{37}{2160} \ln 2 - \frac{7}{960} \ln 3 \\
& \quad - \frac{1}{240\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)} \\
& = \frac{61\pi}{6912\sqrt{3}} - \frac{35\pi}{6912} + \frac{1}{768} \ln 3 - \frac{7}{432} \ln 2 \\
& \quad + \frac{5}{384\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-8)(12n-7)\cdots(12n-4)} \\
& = - \frac{19\pi}{3840\sqrt{3}} + \frac{11\pi}{3840} + \frac{1}{90} \ln 2 + \frac{17}{3840} \ln 3 \\
& \quad - \frac{19}{1152\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-8)(12n-7)(12n-6)\cdots(12n-3)} \\
& = - \frac{19\pi}{11520\sqrt{3}} + \frac{11\pi}{11520} - \frac{1}{120} \ln 2 - \frac{19}{3840} \ln 3 \\
& \quad + \frac{17}{1152\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-7)(12n-6)(12n-5)\cdots(12n-2)} \\
& = \frac{35\pi}{6912\sqrt{3}} - \frac{101\pi}{34560} + \frac{1}{768} \ln 3 + \frac{19}{2160} \ln 2 \\
& \quad - \frac{19}{1920\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$



$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-6)(12n-5)(12n-4)\cdots(12n-1)} \\
&= -\frac{71\pi}{17280\sqrt{3}} + \frac{41\pi}{17280} + \frac{1}{480} \ln 3 - \frac{17}{2160} \ln 2 \\
&\quad + \frac{1}{240\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)} \\
&= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)} \right. \\
&\quad \left. - \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)} \right] \\
&= -\frac{607\pi}{207360\sqrt{3}} + \frac{119\pi}{69120} + \frac{1}{180} \ln 2 \\
&\quad - \frac{11}{7680} \ln 3 - \frac{11}{3840\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)} \\
&= \frac{476\pi}{207360\sqrt{3}} - \frac{274\pi}{207360} - \frac{59}{12960} \ln 2 - \frac{1}{1920} \ln 3 \\
&\quad + \frac{17}{3456\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-8)(12n-7)\cdots(12n-3)} \\
&= -\frac{19\pi}{34560\sqrt{3}} + \frac{11\pi}{34560} + \frac{7}{2160} \ln 2 + \frac{1}{640} \ln 3 \\
&\quad - \frac{1}{192\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-8)(12n-7)(12n-6)\cdots(12n-2)}$$

$$= -\frac{29\pi}{25920\sqrt{3}} + \frac{67\pi}{103680} - \frac{37}{12960} \ln 2 - \frac{1}{960} \ln 3 \\ + \frac{71}{17280\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-7)(12n-6)(12n-5)\cdots(12n-1)} \\ = \frac{317\pi}{207360\sqrt{3}} - \frac{61\pi}{69120} + \frac{1}{360} \ln 2 - \frac{1}{7680} \ln 3 \\ - \frac{\sqrt{3}}{1280} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)} \\ = \frac{1}{7} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)} \right. \\ \left. - \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)} \right] \\ = -\frac{361\pi}{483840\sqrt{3}} + \frac{631\pi}{1451520} + \frac{131}{90720} \ln 2 - \frac{1}{7680} \ln 3 \\ - \frac{269}{241920\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)} \\ = \frac{59\pi}{145152\sqrt{3}} - \frac{17\pi}{72576} - \frac{101}{90720} \ln 2 - \frac{1}{3360} \ln 3 \\ + \frac{5}{3456\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-8)(12n-7)\cdots(12n-2)} \\ = \frac{59\pi}{725760\sqrt{3}} - \frac{17\pi}{362880} + \frac{79}{90720} \ln 2 + \frac{1}{2688} \ln 3$$

$$-\frac{161}{120960\sqrt{3}} \ln(2+\sqrt{3}),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(12n-8)(12n-7)(12n-6)\cdots(12n-1)} \\ &= -\frac{61\pi}{161280\sqrt{3}} + \frac{317\pi}{1451520} - \frac{73}{90720} \ln 2 - \frac{1}{7680} \ln 3 \\ & \quad + \frac{223}{241920\sqrt{3}} \ln(2+\sqrt{3}); \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)} \\ &= \frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)} \right. \\ & \quad \left. - \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)} \right] \\ &= -\frac{239\pi}{1658880\sqrt{3}} + \frac{971\pi}{11612160} + \frac{29}{90720} \ln 2 \\ & \quad + \frac{3}{143360} \ln 3 - \frac{619}{1935360\sqrt{3}} \ln(2+\sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)} \\ &= \frac{59\pi}{1451520\sqrt{3}} - \frac{17\pi}{725760} - \frac{1}{4032} \ln 2 - \frac{3}{35840} \ln 3 \\ & \quad + \frac{1}{2880\sqrt{3}} \ln(2+\sqrt{3}); \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-8)(12n-7)\cdots(12n-1)} \\ &= \frac{667\pi}{11612160\sqrt{3}} - \frac{11\pi}{331776} + \frac{19}{90720} \ln 2 + \frac{9}{143360} \ln 3 \\ & \quad - \frac{109}{387072\sqrt{3}} \ln(2+\sqrt{3}), \end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)} \\
&= \frac{1}{9} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)} \right. \\
&\quad \left. - \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)} \right] \\
&= -\frac{143\pi}{6967296\sqrt{3}} + \frac{1243\pi}{104509440} + \frac{103}{1632960} \ln 2 \\
&\quad + \frac{1}{86016} \ln 3 - \frac{1291}{17418240\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)} \\
&= -\frac{13\pi}{6967296\sqrt{3}} + \frac{113\pi}{104509440} - \frac{83}{1632960} \ln 2 \\
&\quad - \frac{1}{61440} \ln 3 + \frac{1217}{17418240\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-1)} \\
&= \frac{1}{10} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)} \right. \\
&\quad \left. - \frac{n}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)} \right] \\
&= -\frac{13\pi}{6967296\sqrt{3}} + \frac{113\pi}{104509440} + \frac{31}{2721600} \ln 2 \\
&\quad + \frac{1}{358400} \ln 3 - \frac{209}{14515200\sqrt{3}} \ln(2+\sqrt{3}).
\end{aligned}$$

$$(四) \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-7)}$$

$$\begin{aligned}
&= \frac{1}{96} \sum_{n=1}^{\infty} \left( \frac{11}{12n-11} - \frac{18}{12n-9} + \frac{7}{12n-7} \right) \\
&= \frac{\pi}{192\sqrt{3}} + \frac{\pi}{128} + \frac{3}{128} \ln 3 + \frac{1}{96\sqrt{3}} \ln (2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-7)(12n-5)} \\
&= \frac{1}{96} \sum_{n=1}^{\infty} \left( \frac{9}{12n-9} - \frac{14}{12n-7} + \frac{5}{12n-5} \right) \\
&= \frac{19\pi}{768\sqrt{3}} - \frac{29\pi}{2304} - \frac{3}{256} \ln 3 + \frac{\sqrt{3}}{128} \ln (2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{n}{(12n-7)(12n-5)(12n-3)} \\
&= \frac{1}{96} \sum_{n=1}^{\infty} \left( \frac{7}{12n-7} - \frac{10}{12n-5} + \frac{3}{12n-3} \right) \\
&= -\frac{17\pi}{768\sqrt{3}} + \frac{31\pi}{2304} - \frac{1}{256} \ln 3 \\
&\quad + \frac{1}{128\sqrt{3}} \ln (2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-3)(12n-1)} \\
&= \frac{1}{96} \sum_{n=1}^{\infty} \left( \frac{5}{12n-5} - \frac{6}{12n-3} + \frac{1}{12n-1} \right) \\
&= \frac{\pi}{192\sqrt{3}} - \frac{\pi}{384} + \frac{1}{128} \ln 3 - \frac{1}{96\sqrt{3}} \ln (2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{n}{(12n-3)(12n-1)(12n+1)} \\
&= \frac{1}{96} \sum_{n=1}^{\infty} \left( \frac{3}{12n-3} - \frac{2}{12n-1} - \frac{1}{12n+1} \right)
\end{aligned}$$

$$= \frac{\pi}{768\sqrt{3}} - \frac{\pi}{2304} - \frac{1}{256} \ln 3 - \frac{1}{128\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{96}.$$

由此顺次求得

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-3)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-9)(12n-7)} \right. \\ & \quad \left. - \frac{2n}{(12n-5)(12n-3)(12n-1)} \right] \\ &= 2 \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-7)} \\ & \quad - \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-7)} \\ & \quad - 2 \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-3)(12n-1)} \\ &= \frac{\pi}{96} + \frac{1}{24\sqrt{3}} \ln(2 + \sqrt{3}), \\ & \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)} \\ &= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-9)(12n-7)} \right. \\ & \quad \left. - \frac{n}{(12n-9)(12n-7)(12n-5)} \right] \\ &= -\frac{5\pi}{1536\sqrt{3}} + \frac{47\pi}{13824} + \frac{3}{512} \ln 3 \\ & \quad - \frac{5}{2304\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-7)(12n-5)(12n-3)}$$

$$= \frac{\pi}{128\sqrt{3}} - \frac{5\pi}{1152} - \frac{1}{768} \ln 3$$

$$+ \frac{1}{384\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-7)(12n-5)(12n-3)(12n-1)}$$

$$= -\frac{7\pi}{1536\sqrt{3}} + \frac{37\pi}{13824} - \frac{1}{512} \ln 3$$

$$+ \frac{7}{2304\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-3)(12n-1)(12n+1)}$$

$$= \frac{\pi}{1536\sqrt{3}} - \frac{5\pi}{13824} + \frac{1}{512} \ln 3$$

$$- \frac{1}{2304\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{576}.$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-9)(12n-7)(12n-5)} \right.$$

$$\left. - \frac{2n}{(12n-5)(12n-3)(12n-1)(12n+1)} \right]$$

$$= 2 \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$- \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)}$$

$$\begin{aligned}
& - 2 \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-3)(12n-1)(12n+1)} \\
& = \frac{5\pi}{3456} + \frac{1}{576\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{288}, \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)(12n-3)} \\
& = \frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)} \right. \\
& \quad \left. - \frac{n}{(12n-9)(12n-7)(12n-5)(12n-3)} \right] \\
& = -\frac{17\pi}{12288\sqrt{3}} + \frac{107\pi}{110592} + \frac{11}{12288} \ln 3 \\
& \quad - \frac{11}{18432\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)} \\
& = \frac{19\pi}{12288\sqrt{3}} - \frac{97\pi}{110592} + \frac{1}{12288} \ln 3 \\
& \quad - \frac{1}{18432\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-7)(12n-5)(12n-3)(12n-1)(12n+1)} \\
& = -\frac{\pi}{1536\sqrt{3}} + \frac{7\pi}{18432} - \frac{1}{2048} \ln 3 \\
& \quad + \frac{1}{2304\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{4608}, \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)} \\
& = \frac{1}{10} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-9)(12n-7)\cdots(12n-3)} \right.
\end{aligned}$$



$$\begin{aligned}
& - \frac{n}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)} \Big] \\
& = - \frac{\pi\sqrt{3}}{10240} + \frac{17\pi}{92160} + \frac{1}{12288} \ln 3 \\
& \quad - \frac{1}{18432\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)} \\
& = \frac{3\pi\sqrt{3}}{40960} - \frac{139\pi}{1105920} + \frac{7}{122880} \ln 3 \\
& \quad - \frac{1}{20480\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{46080},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-7)\cdots(12n+1)} \\
& = \frac{1}{12} \sum_{n=1}^{\infty} \left[ \frac{n}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)} \right. \\
& \quad \left. - \frac{n}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)} \right] \\
& = - \frac{7\pi}{163840\sqrt{3}} + \frac{343\pi}{13271040} + \frac{1}{491520} \ln 3 \\
& \quad - \frac{1}{2211840\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{552960}.
\end{aligned}$$

$$\begin{aligned}
3. \quad \sum_{n=1}^{\infty} & \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{D}{k(n-1)+d} \right. \\
& \quad \left. - \frac{A+D-B}{k(n-1)+e} \right]
\end{aligned}$$

定理11 设  $A$ 、 $B$ 、 $D$ 、 $b$ 、 $d$ 、 $e$  都是正整数, 则

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{D}{k(n-1)+d} \right. \\
& \quad \left. - \frac{A+D-B}{k(n-1)+e} \right] \\
&= AC(a, k) - BC(b, k) + DC(d, k) \\
& \quad - (A+D-B)C(e, k) + \frac{1}{k} \ln \frac{b^B e^{A+D-B}}{a^A d^D}.
\end{aligned}$$

**证明** 根据定理2, 有

$$\begin{aligned}
& \sum_{i=1}^{\infty} \left[ \frac{A}{k(i-1)+a} - \frac{B}{k(i-1)+b} + \frac{D}{k(i-1)+d} \right. \\
& \quad \left. - \frac{A+D-B}{k(i-1)+e} \right] \\
&= AC(a, k) - BC(b, k) + DC(d, k) \\
& \quad - (A+D-B)C(e, k) \\
& \quad + \frac{1}{k} \ln \frac{(nk-k+a)^A (nk-k+d)^D}{(nk-k+b)^B (nk-k+e)^{A+D-B}} \\
& \quad + \frac{1}{k} \ln \frac{b^B e^{A+D-B}}{a^A d^D} + A\varepsilon_n(a, k) - B\varepsilon_n(b, k) \\
& \quad + D\varepsilon_n(d, k) - (A+D-B)\varepsilon_n(e, k),
\end{aligned}$$

令  $n \rightarrow \infty$  并取极限, 定理即得证

**例1** 求和:

$$\sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)}.$$

**解** 用分项分式方法求得

$$\begin{aligned}
& \frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)} \\
&= \frac{1}{288} \left( \frac{5}{6n-5} - \frac{9}{6n-3} + \frac{3}{6n-1} + \frac{1}{6n+1} \right).
\end{aligned}$$

于定理 11 令  $A=5$ ,  $B=9$ ,  $D=3$ ,  $a=1$ ,  $b=3$ ,  $d=5$ ,  
 $e=7$ ,  $k=6$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)} \\ &= \frac{1}{288} \sum_{n=1}^{\infty} \left( \frac{5}{6n-5} - \frac{9}{6n-3} + \frac{3}{6n-1} + \frac{1}{6n+1} \right) \\ &= \frac{5}{288} C(1,6) - \frac{1}{96} C(1,2) + \frac{1}{96} C(5,6) \\ & \quad + \frac{1}{288} C(7,6) + \frac{1}{1728} \ln \frac{3^9}{5^3 7}. \end{aligned}$$

于命题 2 令  $a=1$ ,  $b=1$ ,  $k=6$ , 则有

$$\begin{aligned} C(7,6) &= C(1,6) + \frac{1}{6} \ln 7 - 1, \\ \frac{1}{288} C(7,6) &= \frac{1}{288} C(1,6) + \frac{1}{1728} \ln 7 - \frac{1}{288}. \end{aligned}$$

$$\begin{aligned} \text{于是} \quad & \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)} \\ &= \frac{1}{48} C(1,6) - \frac{1}{96} C(1,2) + \frac{1}{96} C(5,6) \\ & \quad + \frac{1}{1728} \ln \frac{3^9}{5^3} - \frac{1}{288}. \end{aligned}$$

根据定理 6, 有

$$\begin{aligned} \frac{1}{48} C(1,6) - \frac{1}{144} C(1,2) &= \frac{\pi}{192\sqrt{3}} + \frac{1}{576} \ln 3, \\ \frac{1}{96} C(5,6) - \frac{1}{288} C(1,2) &= -\frac{\pi}{384\sqrt{3}} + \frac{1}{1152} \ln 3 \cdot 5^2. \end{aligned}$$

$$\text{于是} \quad \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-3)(6n-1)(6n+1)}$$

$$= \frac{\pi}{384\sqrt{3}} + \frac{1}{128} \ln 3 - \frac{1}{288}.$$

例2 求和:

$$\sum_{n=1}^{\infty} \frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)}.$$

解 用分项分式方法求得

$$\begin{aligned} & \frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)} \\ &= \frac{1}{216} \left( \frac{25}{6n-5} - \frac{48}{6n-4} + \frac{27}{6n-3} - \frac{4}{6n-2} \right). \end{aligned}$$

于定理11令  $A=25$ ,  $B=48$ ,  $D=27$ ,  $a=1$ ,  $b=2$ ,  $d=3$ ,  $e=4$ ,  $k=6$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)} \\ &= \frac{1}{216} \sum_{n=1}^{\infty} \left( \frac{25}{6n-5} - \frac{48}{6n-4} + \frac{72}{6n-3} - \frac{4}{6n-2} \right) \\ &= \frac{25}{216} C(1,6) - \frac{1}{9} C(1,3) + \frac{1}{24} C(1,2) - \frac{1}{108} C(2,3) \\ & \quad + \frac{1}{1296} \ln \frac{2^{56}}{3^{27}}. \end{aligned}$$

根据定理6, 有

$$\begin{aligned} \frac{25}{216} C(1,6) - \frac{25}{432} C(1,3) &= \frac{25\pi}{1296\sqrt{3}} + \frac{1}{1296} \ln 2^{26}, \\ \frac{23}{648} C(1,2) - \frac{23}{432} C(1,3) &= -\frac{23\pi}{2592\sqrt{3}} \\ & \quad + \frac{1}{2592} \ln \frac{2^{46}}{3^{23}}, \\ \frac{1}{162} C(1,2) - \frac{1}{108} C(2,3) &= \frac{\pi}{648\sqrt{3}} + \frac{1}{648} \ln \frac{1}{3}. \end{aligned}$$

$$\begin{aligned}\text{于是 } \sum_{n=1}^{\infty} \frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)} \\ = \frac{31\pi}{2592\sqrt{3}} + \frac{13}{162} \ln 2 - \frac{1}{32} \ln 3.\end{aligned}$$

仿此求得

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^2}{(6n-4)(6n-3)(6n-2)(6n-1)} \\ = \frac{1}{216} \sum_{n=1}^{\infty} \left( \frac{16}{6n-4} - \frac{27}{6n-3} + \frac{12}{6n-2} - \frac{1}{6n-1} \right) \\ = \frac{7\pi}{2592\sqrt{3}} + \frac{1}{32} \ln 3 - \frac{7}{162} \ln 2.\end{aligned}$$

$$\begin{aligned}\text{于是 } \sum_{n=1}^{\infty} \frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)} \\ = \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)} \right. \\ \left. - \frac{n^2}{(6n-4)(6n-3)(6n-2)(6n-1)} \right] \\ = \frac{\pi}{432\sqrt{3}} + \frac{5}{162} \ln 2 - \frac{1}{64} \ln 3.\end{aligned}$$

仿此求得

$$\begin{aligned}(\text{一}) \sum_{n=1}^{\infty} \frac{n^2}{(8n-7)(8n-6)(8n-5)(8n-4)} \\ = \frac{1}{384} \sum_{n=1}^{\infty} \left( \frac{49}{8n-7} - \frac{108}{8n-6} + \frac{75}{8n-5} - \frac{16}{8n-4} \right) \\ = \frac{31\pi}{768\sqrt{2}} - \frac{67\pi}{3072} + \frac{35}{768} \ln 2 \\ - \frac{13}{768\sqrt{2}} \ln(\sqrt{2}+1),\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(8n-6)(8n-5)(8n-4)(8n-3)} \\
&= \frac{1}{128} \sum_{n=1}^{\infty} \left( \frac{12}{8n-6} - \frac{25}{8n-5} + \frac{16}{8n-4} - \frac{3}{8n-3} \right) \\
&= -\frac{11\pi}{512\sqrt{2}} + \frac{17\pi}{1024} - \frac{11}{256} \ln 2 \\
&\quad + \frac{7}{128\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(8n-5)(8n-4)(8n-3)(8n-2)} \\
&= \frac{1}{384} \sum_{n=1}^{\infty} \left( \frac{25}{8n-5} - \frac{48}{8n-4} + \frac{27}{8n-3} - \frac{4}{8n-2} \right) \\
&= -\frac{\pi}{1536\sqrt{2}} + \frac{\pi}{1024} + \frac{25}{768} \ln 2 \\
&\quad - \frac{13}{384\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(8n-4)(8n-3)(8n-2)(8n-1)} \\
&= \frac{1}{384} \sum_{n=1}^{\infty} \left( \frac{16}{8n-4} - \frac{27}{8n-3} + \frac{12}{8n-2} - \frac{1}{8n-1} \right) \\
&= \frac{7\pi}{768\sqrt{2}} - \frac{19\pi}{3072} - \frac{11}{768} \ln 2 \\
&\quad + \frac{13}{768\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

由此顺次求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \\
&= -\frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(8n-7)(8n-6)(8n-5)(8n-4)} \right.
\end{aligned}$$

$$- \frac{n^2}{(8n-6)(8n-5)(8n-4)(8n-3)} \Bigg\}$$

$$= \frac{95\pi}{6144\sqrt{2}} - \frac{59\pi}{6144} + \frac{17}{768} \ln 2$$

$$- \frac{55}{3072\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)}$$

$$= -\frac{\pi}{192\sqrt{2}} + \frac{\pi}{256} - \frac{29}{1536} \ln 2$$

$$+ \frac{17}{768\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(8n-5)(8n-4)(8n-3)(8n-2)(8n-1)}$$

$$= -\frac{5\pi}{2048\sqrt{2}} + \frac{11\pi}{6144} + \frac{3}{256} \ln 2$$

$$- \frac{13}{1024\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \right.$$

$$\left. - \frac{n^2}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)} \right]$$

$$= \frac{127\pi}{30720\sqrt{2}} - \frac{83\pi}{30720} + \frac{21}{2560} \ln 2$$

$$- \frac{41}{5120\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)}$$

$$\begin{aligned}
&= -\frac{17\pi}{30720\sqrt{2}} + \frac{13\pi}{30720} - \frac{47}{7680} \ln 2 \\
&\quad + \frac{107}{15360\sqrt{2}} \ln(\sqrt{2} + 1); \\
&\sum_{n=1}^{\infty} \frac{n^2}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)} \\
&= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)} \right. \\
&\quad \left. - \frac{n^2}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)} \right] \\
&= \frac{\pi}{1280\sqrt{2}} - \frac{\pi}{1920} + \frac{11}{4608} \ln 2 \\
&\quad - \frac{23}{9216\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

$$\begin{aligned}
(二) \quad &\sum_{n=1}^{\infty} \frac{n^2}{(8n-7)(8n-5)(8n-3)(8n-1)} \\
&= \frac{1}{3072} \sum_{n=1}^{\infty} \left( \frac{49}{8n-7} - \frac{75}{8n-5} + \frac{27}{8n-3} - \frac{1}{8n-1} \right) \\
&= -\frac{13\pi}{6144\sqrt{2}} + \frac{19\pi}{6144} + \frac{1}{128\sqrt{2}} \ln(\sqrt{2} + 1), \\
&\sum_{n=1}^{\infty} \frac{n^2}{(8n-5)(8n-3)(8n-1)(8n+1)} \\
&= \frac{1}{3072} \sum_{n=1}^{\infty} \left( \frac{25}{8n-5} - \frac{27}{8n-3} + \frac{3}{8n-1} - \frac{1}{8n+1} \right) \\
&= \frac{\pi}{512\sqrt{2}} - \frac{7\pi}{6144} + \frac{1}{3072\sqrt{2}} \ln(\sqrt{2} + 1) + \frac{1}{3072}, \\
&\sum_{n=1}^{\infty} \frac{n^2}{(8n-7)(8n-5)(8n-3)(8n-1)(8n+1)}
\end{aligned}$$



$$\begin{aligned}
&= \frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(8n-7)(8n-5)(8n-3)(8n-1)} \right. \\
&\quad \left. - \frac{n^2}{(8n-5)(8n-3)(8n-1)(8n+1)} \right] \\
&= -\frac{25\pi}{49152\sqrt{2}} + \frac{13\pi}{24576} + \frac{23}{24576\sqrt{2}} \ln(\sqrt{2}+1) \\
&\quad - \frac{1}{24576}.
\end{aligned}$$

$$\begin{aligned}
(\text{三}) \quad &\sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)(12n-8)} \\
&= \frac{1}{864} \sum_{n=1}^{\infty} \left( \frac{121}{12n-11} - \frac{300}{12n-10} + \frac{243}{12n-9} - \frac{64}{12n-8} \right) \\
&= -\frac{601\pi}{20736\sqrt{3}} + \frac{485\pi}{20736} + \frac{41}{864} \ln 2 - \frac{9}{256} \ln 3 \\
&\quad + \frac{121}{3456\sqrt{3}} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^2}{(12n-10)(12n-9)(12n-8)(12n-7)} \\
&= \frac{1}{864} \sum_{n=1}^{\infty} \left( \frac{100}{12n-10} - \frac{243}{12n-9} + \frac{192}{12n-8} - \frac{49}{12n-7} \right) \\
&= \frac{71\pi}{2304\sqrt{3}} - \frac{341\pi}{20736} + \frac{9}{256} \ln 3 - \frac{169}{2592} \ln 2 \\
&\quad + \frac{49}{3456\sqrt{3}} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-8)(12n-7)(12n-6)} \\
&= \frac{1}{288} \sum_{n=1}^{\infty} \left( \frac{27}{12n-9} - \frac{64}{12n-8} + \frac{49}{12n-7} - \frac{12}{12n-6} \right)
\end{aligned}$$

$$= -\frac{211\pi}{6912\sqrt{3}} + \frac{125\pi}{6912} + \frac{17}{288} \ln 2 - \frac{5}{768} \ln 3$$

$$- \frac{49}{1152\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$= \frac{1}{864} \sum_{n=1}^{\infty} \left( \frac{64}{12n-8} - \frac{147}{12n-7} + \frac{108}{12n-6} - \frac{25}{12n-5} \right)$$

$$= \frac{215\pi}{10368\sqrt{3}} - \frac{61\pi}{5184} - \frac{25}{864} \ln 2 - \frac{1}{64} \ln 3$$

$$+ \frac{43}{864\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$= \frac{1}{864} \sum_{n=1}^{\infty} \left( \frac{49}{12n-7} - \frac{108}{12n-6} + \frac{75}{12n-5} - \frac{16}{12n-4} \right)$$

$$= \frac{47\pi}{10368\sqrt{3}} - \frac{13\pi}{5184} + \frac{13}{864} \ln 2 + \frac{1}{64} \ln 3$$

$$- \frac{31}{864\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-6)(12n-5)(12n-4)(12n-3)}$$

$$= \frac{1}{864} \sum_{n=1}^{\infty} \left( \frac{36}{12n-6} - \frac{75}{12n-5} + \frac{48}{12n-4} - \frac{9}{12n-3} \right)$$

$$= -\frac{91\pi}{6912\sqrt{3}} + \frac{53\pi}{6912} - \frac{5}{288} \ln 2 - \frac{1}{256} \ln 3$$

$$+ \frac{25}{1152\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-5)(12n-4)(12n-3)(12n-2)}$$

$$\begin{aligned}
&= \frac{1}{864} \sum_{n=1}^{\infty} \left( \frac{25}{12n-5} - \frac{48}{12n-4} + \frac{27}{12n-3} - \frac{4}{12n-2} \right) \\
&= \frac{5\pi}{768\sqrt{3}} - \frac{77\pi}{20736} + \frac{37}{2592} \ln 2 - \frac{1}{256} \ln 3 \\
&\quad - \frac{25}{3456\sqrt{3}} \ln(2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{n^2}{(12n-4)(12n-3)(12n-2)(12n-1)} \\
&= \frac{1}{864} \sum_{n=1}^{\infty} \left( \frac{16}{12n-4} - \frac{27}{12n-3} + \frac{12}{12n-2} - \frac{1}{12n-1} \right) \\
&= -\frac{49\pi}{20736\sqrt{3}} + \frac{29\pi}{20736} + \frac{1}{256} \ln 3 - \frac{5}{864} \ln 2 \\
&\quad - \frac{1}{3456\sqrt{3}} \ln(2 + \sqrt{3}).
\end{aligned}$$

由此顺次推得

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(12n-11)(12n-10)(12n-9)(12n-8)} \right. \\
&\quad \left. - \frac{4n^2}{(12n-5)(12n-4)(12n-3)(12n-2)} \right] \\
&= 4 \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)(12n-8)} \\
&\quad - 4 \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)} \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)} \\
&\quad - 4 \sum_{n=1}^{\infty} \frac{n^2}{(12n-5)(12n-4)(12n-3)(12n-2)}
\end{aligned}$$

$$= -\frac{13\pi}{324\sqrt{3}} + \frac{77\pi}{2592} - \frac{11}{324} \ln 2$$

$$+ \frac{25}{432\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-10)(12n-9)(12n-8)} \right.$$

$$\left. - \frac{n^2}{(12n-10)(12n-9)(12n-8)(12n-7)} \right]$$

$$= -\frac{155\pi}{10368\sqrt{3}} + \frac{413\pi}{41472} + \frac{73}{2592} \ln 2 - \frac{9}{512} \ln 3$$

$$+ \frac{1}{192\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)}$$

$$= \frac{159\pi}{10368\sqrt{3}} - \frac{179\pi}{20736} + \frac{1}{96} \ln 3 - \frac{161}{5184} \ln 2$$

$$+ \frac{49}{3456\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-8)(12n-7)(12n-6)(12n-5)}$$

$$= -\frac{1063\pi}{82944\sqrt{3}} + \frac{619\pi}{82944} + \frac{19}{864} \ln 2 + \frac{7}{3072} \ln 3$$

$$- \frac{319}{13824\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-8)(12n-7)(12n-6)(12n-5)(12n-4)}$$

$$= \frac{7\pi}{1728\sqrt{3}} - \frac{\pi}{432} - \frac{19}{1728} \ln 2 - \frac{1}{128} \ln 3$$

$$+ \frac{37}{1728\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(12n-7)(12n-6)(12n-5)(12n-4)(12n-3)} \\ &= \frac{367\pi}{82944\sqrt{3}} - \frac{211\pi}{82944} + \frac{7}{864} \ln 2 + \frac{5}{1024} \ln 3 \\ & \quad - \frac{199}{13824\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(12n-6)(12n-5)(12n-4)(12n-3)(12n-2)} \\ &= -\frac{17\pi}{3456\sqrt{3}} + \frac{59\pi}{20736} - \frac{41}{5184} \ln 2 \\ & \quad + \frac{25}{3456\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)} \\ &= \frac{23\pi}{10368\sqrt{3}} - \frac{53\pi}{41472} + \frac{13}{2592} \ln 2 - \frac{1}{512} \ln 3 \\ & \quad - \frac{1}{576\sqrt{3}} \ln(2 + \sqrt{3}). \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)} \right. \\ & \quad \left. - \frac{4n^2}{(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)} \right] \\ &= 4 \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)} \end{aligned}$$

$$\begin{aligned}
& - 4 \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)} \\
& + \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)} \\
& - 4 \sum_{n=1}^{\infty} \frac{n^2}{(12n-5)(12n-4)(12n-3)\cdots(12n-1)} \\
& = -\frac{5\pi}{324\sqrt{3}} + \frac{53\pi}{5184} - \frac{1}{108} \ln 2 \\
& \quad + \frac{1}{72\sqrt{3}} \ln(2 + \sqrt{3}); \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)} \\
& = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-7)} \right. \\
& \quad \left. - \frac{n^2}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-6)} \right] \\
& = -\frac{157\pi}{25920\sqrt{3}} + \frac{257\pi}{69120} + \frac{307}{25920} \ln 2 \\
& \quad - \frac{43}{7680} \ln 3 - \frac{31}{17280\sqrt{3}} \ln(2 + \sqrt{3}); \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)} \\
& = \frac{467\pi}{82944\sqrt{3}} - \frac{89\pi}{27648} + \frac{5}{3072} \ln 3 - \frac{55}{5184} \ln 2 \\
& \quad + \frac{103}{13824\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-8)(12n-7)\cdots(12n-4)}
\end{aligned}$$

$$= -\frac{1399\pi}{414720\sqrt{3}} + \frac{811\pi}{414720} + \frac{19}{2880} \ln 2$$

$$+ \frac{31}{15360} \ln 3 - \frac{41}{4608\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-8)(12n-7)(12n-6)\cdots(12n-3)}$$

$$= -\frac{31\pi}{414720\sqrt{3}} + \frac{19\pi}{414720} - \frac{11}{2880} \ln 2 - \frac{13}{5120} \ln 3$$

$$+ \frac{11}{1536\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-7)(12n-6)(12n-5)\cdots(12n-2)}$$

$$= \frac{155\pi}{82944\sqrt{3}} - \frac{149\pi}{138240} + \frac{83}{25920} \ln 2 + \frac{1}{1024} \ln 3$$

$$- \frac{299}{69120\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-6)(12n-5)(12n-4)\cdots(12n-1)}$$

$$= -\frac{37\pi}{25920\sqrt{3}} + \frac{19\pi}{23040} - \frac{67}{25920} \ln 2 + \frac{1}{2560} \ln 3$$

$$+ \frac{31}{17280\sqrt{3}} \ln(2 + \sqrt{3});$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)}$$

$$= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-6)} \right.$$

$$\left. - \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-5)} \right]$$

$$= -\frac{4847\pi}{2488320\sqrt{3}} + \frac{959\pi}{829440} + \frac{97}{25920} \ln 2$$

$$\begin{aligned}
& -\frac{37}{30720} \ln 3 - \frac{71}{46080\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)} \\
& = \frac{3734\pi}{2488320\sqrt{3}} - \frac{1073\pi}{1244160} - \frac{223}{77760} \ln 2 \\
& \quad - \frac{1}{15360} \ln 3 + \frac{113}{41472\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-8)(12n-7)\cdots(12n-3)} \\
& = -\frac{19\pi}{34560\sqrt{3}} + \frac{11\pi}{34560} + \frac{1}{576} \ln 2 + \frac{7}{9216} \ln 3 \\
& \quad - \frac{37}{13824\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-8)(12n-7)(12n-6)\cdots(12n-2)} \\
& = -\frac{403\pi}{1244160\sqrt{3}} + \frac{233\pi}{1244160} - \frac{91}{77760} \ln 2 \\
& \quad - \frac{3}{5120} \ln 3 + \frac{397}{207360\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-7)(12n-6)(12n-5)\cdots(12n-1)} \\
& = \frac{1367\pi}{2488320\sqrt{3}} - \frac{263\pi}{829440} + \frac{5}{5184} \ln 2 + \frac{1}{10240} \ln 3 \\
& \quad - \frac{47}{46080\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)} \\
& = \frac{1}{7} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)} \right]
\end{aligned}$$



$$\begin{aligned}
& - \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-4)} \Big] \\
& = - \frac{8581\pi}{17418240\sqrt{3}} + \frac{5023\pi}{17418240} + \frac{257}{272160} \ln 2 \\
& \quad - \frac{1}{6144} \ln 3 - \frac{1769}{2903040\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)} \\
& = \frac{2551\pi}{8709120\sqrt{3}} - \frac{1469\pi}{8709120} - \frac{179}{272160} \ln 2 \\
& \quad - \frac{19}{161280} \ln 3 + \frac{1}{1296\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-8)(12n-7)\cdots(12n-2)} \\
& = - \frac{281\pi}{8709120\sqrt{3}} + \frac{163\pi}{8709120} + \frac{113}{272160} \ln 2 \\
& \quad + \frac{31}{161280} \ln 3 - \frac{119}{181440\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-8)(12n-7)(12n-6)\cdots(12n-1)} \\
& = - \frac{2173\pi}{17418240\sqrt{3}} + \frac{251\pi}{3483648} - \frac{83}{272160} \ln 2 \\
& \quad - \frac{1}{10240} \ln 3 + \frac{1217}{2903040\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)} \\
& = \frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-4)} \right. \\
& \quad \left. - \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-3)} \right]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{4561\pi}{46448640\sqrt{3}} + \frac{7961\pi}{139345920} + \frac{109}{544320} \ln 2 \\
&\quad - \frac{29}{5160960} \ln 3 - \frac{4009}{23224320\sqrt{3}} \ln(2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)} \\
&= \frac{59\pi}{1451520\sqrt{3}} - \frac{17\pi}{725760} - \frac{73}{544320} \ln 2 \\
&\quad - \frac{5}{129024} \ln 3 + \frac{259}{1451520\sqrt{3}} \ln(2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-8)(12n-7)\cdots(12n-1)} \\
&= \frac{179\pi}{15482880\sqrt{3}} - \frac{929\pi}{139345920} + \frac{7}{77760} \ln 2 \\
&\quad + \frac{187}{5160960} \ln 3 - \frac{3121}{23224320\sqrt{3}} \ln(2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)} \\
&= \frac{1}{9} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-3)} \right. \\
&\quad \left. - \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-2)} \right] \\
&= -\frac{6449\pi}{418037760\sqrt{3}} + \frac{2245\pi}{250822656} + \frac{91}{2449440} \ln 2 \\
&\quad + \frac{19}{5160960} \ln 3 - \frac{8153}{209018880\sqrt{3}} \ln(2 + \sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)} \\
&= \frac{193\pi}{59719680\sqrt{3}} - \frac{467\pi}{250822656} - \frac{61}{2449440} \ln 2
\end{aligned}$$

$$\begin{aligned}
& -\frac{43}{5160960} \ln 3 + \frac{1453}{41803776\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-1)} \\
& = \frac{1}{10} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-10)(12n-9)\cdots(12n-2)} \right. \\
& \quad \left. - \frac{n^2}{(12n-10)(12n-9)(12n-8)\cdots(12n-1)} \right] \\
& = -\frac{13\pi}{6967296\sqrt{3}} + \frac{113\pi}{104509440} + \frac{19}{3061800} \ln 2 \\
& \quad + \frac{31}{25804800} \ln 3 - \frac{7709}{1045094400\sqrt{3}} \ln(2 + \sqrt{3}).
\end{aligned}$$

$$\begin{aligned}
(四) \quad & \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-9)(12n-7)(12n-5)} \\
& = \frac{1}{6912} \sum_{n=1}^{\infty} \left( \frac{121}{12n-11} - \frac{243}{12n-9} + \frac{147}{12n-7} - \frac{25}{12n-5} \right) \\
& = -\frac{17\pi}{18432\sqrt{3}} + \frac{343\pi}{165888} + \frac{9}{2048} \ln 3 \\
& \quad - \frac{1}{27648\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-7)(12n-5)(12n-3)} \\
& = \frac{1}{2304} \sum_{n=1}^{\infty} \left( \frac{27}{12n-9} - \frac{49}{12n-7} + \frac{25}{12n-5} - \frac{3}{12n-3} \right) \\
& = \frac{37\pi}{9216\sqrt{3}} - \frac{59\pi}{27648} - \frac{1}{768} \ln 3 \\
& \quad + \frac{1}{384\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-7)(12n-5)(12n-3)(12n-1)} \\
&= \frac{1}{6912} \sum_{n=1}^{\infty} \left( \frac{49}{12n-7} - \frac{75}{12n-5} + \frac{27}{12n-3} - \frac{1}{12n-1} \right) \\
&= -\frac{41\pi}{18432\sqrt{3}} + \frac{223\pi}{165888} - \frac{1}{2048} \ln 3 \\
&\quad + \frac{25}{27648\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-5)(12n-3)(12n-1)(12n+1)} \\
&= \frac{1}{6912} \sum_{n=1}^{\infty} \left( \frac{25}{12n-5} - \frac{27}{12n-3} + \frac{3}{12n-1} - \frac{1}{12n+1} \right) \\
&= \frac{7\pi}{18432\sqrt{3}} - \frac{31\pi}{165888} + \frac{1}{2048} \ln 3 \\
&\quad - \frac{23}{27648\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{6912}.
\end{aligned}$$

由此顺次求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(6n-5)(6n-3)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(12n-11)(12n-9)(12n-7)(12n-5)} \right. \\
&\quad \left. - \frac{4n^2}{(12n-5)(12n-3)(12n-1)(12n+1)} \right] \\
&= 4 \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-9)(12n-7)(12n-5)} \\
&\quad - 4 \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-7)(12n-5)} \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-7)(12n-5)}
\end{aligned}$$

$$\begin{aligned}
& - 4 \sum_{n=1}^{\infty} \frac{n^2}{(12n-5)(12n-3)(12n-1)(12n+1)} \\
& = \frac{31\pi}{20736} + \frac{23}{3456\sqrt{3}} \ln(2+\sqrt{3}) - \frac{1}{1728}, \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-9)(12n-7)(12n-5)(12n-3)} \\
& = \frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-9)(12n-7)(12n-5)} \right. \\
& \quad \left. - \frac{n^2}{(12n-9)(12n-7)(12n-5)(12n-3)} \right] \\
& = -\frac{91\pi}{147456\sqrt{3}} + \frac{697\pi}{1327104} + \frac{35}{49152} \ln 3 \\
& \quad - \frac{73}{221184\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)} \\
& = \frac{115\pi}{147456\sqrt{3}} - \frac{577\pi}{1327104} - \frac{5}{49152} \ln 3 \\
& \quad + \frac{47}{221184\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-7)(12n-5)(12n-3)(12n-1)(12n+1)} \\
& = -\frac{\pi}{3072\sqrt{3}} + \frac{127\pi}{663552} - \frac{1}{8192} \ln 3 \\
& \quad + \frac{1}{4608\sqrt{3}} \ln(2+\sqrt{3}) - \frac{1}{55296}, \\
& \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)}
\end{aligned}$$

$$= \frac{1}{10} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-9)(12n-7)\cdots(12n-3)} - \frac{n^2}{(12n-9)(12n-7)(12n-5)(12n-3)(12n-1)} \right]$$

$$= -\frac{103\pi}{737280\sqrt{3}} + \frac{637\pi}{6635520} + \frac{1}{12288} \ln 3 - \frac{1}{18432\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)}$$

$$= \frac{163\pi}{1474560\sqrt{3}} - \frac{277\pi}{4423680} + \frac{1}{491520} \ln 3 - \frac{1}{2211840\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{552960},$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-9)(12n-7)\cdots(12n+1)}$$

$$= \frac{1}{12} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(12n-11)(12n-9)(12n-7)\cdots(12n-1)} - \frac{n^2}{(12n-9)(12n-7)(12n-5)\cdots(12n+1)} \right]$$

$$= -\frac{41\pi}{1966080\sqrt{3}} + \frac{421\pi}{31850495} + \frac{13}{1966080} \ln 3 - \frac{119}{26542080\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{6635520}.$$

例3 求和  $(-1)^{n-1} \frac{1}{(5n-4)(5n-3)(5n-2)(5n-1)}$

解 应用分项分式方法求得

$$\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)}$$

$$= \frac{1}{6} \left( \frac{1}{10n-9} - \frac{3}{10n-8} + \frac{3}{10n-7} - \frac{1}{10n-6} \right).$$

于定理11令  $A=1, B=3, D=3, a=1, b=2, d=3, e=4, k=10$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)} \\ &= \frac{1}{6} \sum_{n=1}^{\infty} \left( \frac{1}{10n-9} - \frac{3}{10n-8} + \frac{3}{10n-7} - \frac{1}{10n-6} \right) \\ &= \frac{1}{6} C(1,10) - \frac{1}{4} C(1,5) + \frac{1}{2} C(3,10) - \frac{1}{12} C(2,5) \\ & \quad + \frac{1}{60} \ln \frac{2^5}{3^3}. \end{aligned}$$

根据定理7, 有

$$\begin{aligned} \frac{1}{6} C(1,10) - \frac{1}{12} C(1,5) &= \frac{1}{12} A(1,5), \\ \frac{1}{3} C(3,10) - \frac{1}{6} C(1,5) &= -\frac{1}{6} A(1,5) + \frac{1}{6} A(3,5) \\ & \quad + \frac{1}{30} \ln 6, \end{aligned}$$

$$\begin{aligned} \frac{1}{6} C(3,10) - \frac{1}{12} C(2,5) \\ &= -\frac{1}{12} A(1,5) + \frac{1}{12} A(2,5) + \frac{1}{12} A(3,5) + \frac{1}{60} \ln 6. \end{aligned}$$

$$\begin{aligned} \text{于是 } \sum_{n=1}^{\infty} \frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)} \\ &= -\frac{1}{6} A(1,5) + \frac{1}{12} A(2,5) + \frac{1}{4} A(3,5) + \frac{2}{15} \ln 2 \\ &= 0.0417137228\cdots. \end{aligned}$$

仿此求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(10n-4)(10n-3)(10n-2)(10n-1)} \\
&= \frac{1}{6} \sum_{n=1}^{\infty} \left( \frac{1}{10n-4} - \frac{3}{10n-3} + \frac{3}{10n-2} - \frac{1}{10n-1} \right) \\
&= -\frac{1}{12} A(1,5) + \frac{1}{3} A(2,5) - \frac{1}{12} A(4,5) - \frac{1}{15} \ln 2 \\
&= 0.00034412169\cdots.
\end{aligned}$$

于是

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(5n-4)(5n-3)(5n-2)(5n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)} \right. \\
&\quad \left. - \frac{1}{(10n-4)(10n-3)(10n-2)(10n-1)} \right] \\
&= -\frac{1}{12} A(1,5) - \frac{1}{4} A(2,5) + \frac{1}{4} A(3,5) \\
&\quad + \frac{1}{12} A(4,5) + \frac{1}{5} \ln 2 \\
&= 0.0413696011\cdots.
\end{aligned}$$

$$\begin{aligned}
4. \quad \sum_{n=1}^{\infty} & \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{D}{k(n-1)+d} \right. \\
& \left. - \frac{E}{k(n-1)+e} + \frac{B+E-A-D}{k(n-1)+f} \right]
\end{aligned}$$

**定理12** 设  $A, B, D, E, b, d, e, f$  都是正整数, 则

$$\sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{D}{k(n-1)+d} \right.$$



$$\begin{aligned}
& \left[ -\frac{E}{k(n-1)+e} + \frac{B+E-A-D}{k(n-1)+f} \right] \\
& = AC(a, k) - BC(b, k) + DC(d, k) - EC(e, k) \\
& \quad + (B+E-A-D)C(f, k) \\
& \quad + \frac{1}{k} \ln \frac{b^B e^E}{a^A d^D f^{B+E-A-D}}.
\end{aligned}$$

**证明** 根据定理 2, 有

$$\begin{aligned}
& \sum_{i=1}^{\infty} \left[ \frac{A}{k(i-1)+a} - \frac{B}{k(i-1)+b} + \frac{D}{k(i-1)+d} \right. \\
& \quad \left. - \frac{E}{k(i-1)+e} + \frac{B+E-A-D}{k(i-1)+f} \right] \\
& = AC(a, k) - BC(b, k) + DC(d, k) - EC(e, k) \\
& \quad + (B+E-A-D)C(f, k) \\
& \quad + \frac{1}{k} \ln \frac{(nk-k+a)^A (nk-k+d)^D (nk-k+f)^{B+E-A-D}}{(nk-k+b)^B (nk-k+e)^E} \\
& \quad + \frac{1}{k} \ln \frac{b^B e^E}{a^A d^D f^{B+E-A-D}} + A\varepsilon_n(a, k) \\
& \quad - B\varepsilon_n(b, k) + D\varepsilon_n(d, k) - E\varepsilon_n(e, k) \\
& \quad + (B+E-A-D)\varepsilon_n(f, k).
\end{aligned}$$

令  $n \rightarrow \infty$  并取极限, 定理即得证.

**例1** 求和:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(5n-4)(5n-3)(5n-2)(5n-1)5n}.$$

**解** 用分项分式方法求得

$$\frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)(10n-5)}$$

$$= \frac{1}{24} \left( \frac{1}{10n-9} - \frac{4}{10n-8} + \frac{6}{10n-7} - \frac{4}{10n-6} + \frac{1}{10n-5} \right).$$

于定理12令  $A=1$ ,  $B=4$ ,  $D=6$ ,  $E=4$ ,  $a=1$ ,  $b=2$ ,  
 $d=3$ ,  $e=4$ ,  $f=5$ ,  $k=10$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)(10n-5)} \\ &= \frac{1}{24} \sum_{n=1}^{\infty} \left( \frac{1}{10n-9} - \frac{4}{10n-8} + \frac{6}{10n-7} - \frac{4}{10n-6} + \frac{1}{10n-5} \right) \\ &= \frac{1}{24} C(1,10) - \frac{1}{12} C(1,5) + \frac{1}{4} C(3,10) - \frac{1}{12} C(2,5) \\ & \quad + \frac{1}{120} C(1,2) + \frac{1}{240} \ln \frac{2^{12}}{3^6 5}. \end{aligned}$$

根据定理7与6, 有

$$\begin{aligned} & \frac{1}{24} C(1,10) - \frac{1}{48} C(1,5) = \frac{1}{48} A(1,5), \\ & \frac{1}{8} C(3,10) - \frac{1}{16} C(1,5) = -\frac{1}{16} A(1,5) + \frac{1}{16} A(3,5) \\ & \quad + \frac{1}{80} \ln 6, \\ & \frac{1}{8} C(3,10) - \frac{1}{16} C(2,5) \\ &= -\frac{1}{16} A(1,5) + \frac{1}{16} A(2,5) + \frac{1}{16} A(3,5) + \frac{1}{80} \ln 6, \\ & \frac{1}{120} C(1,2) - \frac{1}{48} C(2,5) \\ &= -\frac{1}{192} A(1,5) + \frac{1}{96} A(2,5) - \frac{1}{192} A(4,5) \\ & \quad + \frac{1}{960} \ln \frac{2^2}{5}. \end{aligned}$$

$$\begin{aligned}
\text{于是 } & \sum_{n=1}^{\infty} \frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)(10n-5)} \\
&= -\frac{7}{64} A(1,5) + \frac{7}{96} A(2,5) + \frac{1}{8} A(3,5) \\
&\quad - \frac{1}{192} A(4,5) + \frac{37}{480} \ln 2 - \frac{1}{192} \ln 5 \\
&= 0.0083363022\cdots.
\end{aligned}$$

仿此求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(10n-4)(10n-3)(10n-2)(10n-1)10n} \\
&= \frac{1}{24} \sum_{n=1}^{\infty} \left( \frac{1}{10n-4} - \frac{4}{10n-3} + \frac{6}{10n-2} - \frac{4}{10n-1} + \frac{1}{10n} \right) \\
&= -\frac{5}{192} A(1,5) + \frac{11}{96} A(2,5) + \frac{7}{192} A(4,5) \\
&\quad - \frac{1}{32} \ln 2 - \frac{1}{192} \ln 5 \\
&= 0.000033684414\cdots.
\end{aligned}$$

$$\begin{aligned}
\text{于是 } & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(5n-4)(5n-3)(5n-2)(5n-1)5n} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(10n-9)(10n-8)(10n-7)(10n-6)(10n-5)} \right. \\
&\quad \left. - \frac{1}{(10n-4)(10n-3)(10n-2)(10n-1)10n} \right] \\
&= -\frac{1}{12} A(1,5) - \frac{1}{24} A(2,5) + \frac{1}{8} A(3,5) - \frac{1}{24} A(4,5) \\
&\quad + \frac{13}{120} \ln 2 \\
&= 0.0083026178\cdots.
\end{aligned}$$

例2 求和:

$$\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)}$$

解 用分项分式方法求得

$$\begin{aligned} & \frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)} \\ &= \frac{1}{69120} \left( \frac{22}{12n-11} - \frac{105}{12n-7} + \frac{100}{12n-5} - \frac{12}{12n-1} \right. \\ & \quad \left. - \frac{5}{12n+1} \right). \end{aligned}$$

于定理12令  $A=22, B=105, D=100, E=12, a=1,$   
 $b=5, d=7, e=11, f=13, k=12$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)} \\ &= \frac{1}{69120} \sum_{n=1}^{\infty} \left( \frac{22}{12n-11} - \frac{105}{12n-7} + \frac{100}{12n-5} - \frac{12}{12n-1} \right. \\ & \quad \left. - \frac{5}{12n+1} \right) \\ &= \frac{11}{34560} C(1,12) - \frac{7}{4608} C(5,12) + \frac{5}{3456} C(7,12) \\ & \quad - \frac{1}{5760} C(11,12) - \frac{1}{13824} C(13,12) \\ & \quad + \frac{1}{829440} \ln \frac{5^{105} 11^{12} 13^5}{7^{100}}. \end{aligned}$$

于命题2令  $a=1, b=1, k=12$ , 则有

$$\begin{aligned} C(13,12) &= C(1,12) + \frac{1}{12} \ln 13 - 1, \\ -\frac{1}{13824} C(13,12) &= -\frac{1}{13824} C(1,12) \end{aligned}$$

$$-\frac{1}{165888} \ln 13 + \frac{1}{13824}.$$

$$\begin{aligned} \text{于是 } & \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)} \\ &= \frac{17}{69120} C(1,12) - \frac{7}{4608} C(5,12) + \frac{5}{3456} C(7,12) \\ & - \frac{1}{5760} C(11,12) + \frac{1}{829440} \ln \frac{5^{105} 11^{12}}{7^{100}} + \frac{1}{13824}. \end{aligned}$$

根据定理 6, 有

$$\begin{aligned} & \frac{17}{69120} C(1,12) - \frac{17}{69120} C(5,12) \\ &= \frac{17\pi}{276480\sqrt{3}} - \frac{1}{829440} \ln 5^{17} \\ & \quad + \frac{17}{138240\sqrt{3}} \ln(2 + \sqrt{3}), \\ & \frac{11}{8640} C(7,12) - \frac{11}{8640} C(5,12) \\ &= \frac{11\pi}{34560\sqrt{3}} - \frac{11\pi}{51840} + \frac{1}{103680} \ln \frac{7^{11}}{5^{11}}, \\ & \frac{1}{5760} C(7,12) - \frac{1}{5760} C(11,12) \\ &= \frac{\pi}{23040\sqrt{3}} + \frac{1}{69120} \ln \frac{7}{11} \\ & \quad - \frac{1}{11520\sqrt{3}} \ln(2 + \sqrt{3}). \end{aligned}$$

$$\begin{aligned} \text{于是 } & \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-7)(12n-5)(12n-1)(12n+1)} \\ &= \frac{13\pi}{30720\sqrt{3}} - \frac{11\pi}{51840} + \frac{1}{27648\sqrt{3}} \ln(2 + \sqrt{3}) \end{aligned}$$

$$+ \frac{1}{13824}.$$

例3 求和:

$$\sum_{n=1}^{\infty} \frac{n^3}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}.$$

解 用分项分式方法求得

$$\begin{aligned} & \frac{n^3}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \\ &= \frac{1}{12288} \left( \frac{343}{8n-7} - \frac{864}{8n-6} + \frac{750}{8n-5} - \frac{256}{8n-4} \right. \\ & \quad \left. + \frac{27}{8n-3} \right). \end{aligned}$$

于定理12, 令  $A=343$ ,  $B=864$ ,  $D=750$ ,  $E=256$ ,  $a=1$ ,  $b=2$ ,  $d=3$ ,  $e=4$ ,  $f=5$ ,  $k=8$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^3}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \\ &= \frac{1}{12288} \sum_{n=1}^{\infty} \left( \frac{343}{8n-7} - \frac{864}{8n-6} + \frac{750}{8n-5} - \frac{256}{8n-4} \right. \\ & \quad \left. + \frac{27}{8n-3} \right) \\ &= \frac{343}{12288} C(1,8) - \frac{9}{256} C(1,4) + \frac{125}{2048} C(3,8) \\ & \quad - \frac{1}{192} C(1,2) + \frac{9}{4096} C(5,8) + \frac{1}{98304} \ln \frac{2^{1876}}{3^{760} 5^{27}}. \end{aligned}$$

根据定理6, 有

$$\begin{aligned} & \frac{343}{12288} C(1,8) - \frac{343}{24576} C(1,4) \\ &= \frac{343\pi}{98304\sqrt{2}} + \frac{343}{49152\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

$$\begin{aligned}
& \frac{521}{12288} C(3,8) - \frac{521}{24576} C(1,4) \\
&= \frac{521\pi}{98304\sqrt{2}} - \frac{521\pi}{98304} + \frac{1}{98304} \ln 3^{521} \\
&\quad - \frac{521}{49152\sqrt{2}} \ln(\sqrt{2}+1), \\
& \frac{229}{12288} C(3,8) - \frac{229}{49152} C(1,2) \\
&= \frac{229\pi}{98304\sqrt{2}} - \frac{229\pi}{196608} + \frac{1}{98304} \ln 3^{229} \\
&\quad - \frac{229}{49152\sqrt{2}} \ln(\sqrt{2}+1), \\
& \frac{9}{4096} C(5,8) - \frac{9}{16384} C(1,2) \\
&= -\frac{9\pi}{32768\sqrt{2}} + \frac{9\pi}{65536} + \frac{1}{32768} \ln 5^9 \\
&\quad - \frac{9}{16384\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

于是 
$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^8}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \\
&= \frac{533\pi}{49152\sqrt{2}} - \frac{311\pi}{49152} + \frac{43}{3072} \ln 2 \\
&= \frac{533\pi}{24576\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

仿此求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^8}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)} \\
&= \frac{1}{3072} \sum_{n=1}^{\infty} \left( \frac{54}{8n-6} - \frac{125}{8n-5} + \frac{96}{8n-4} - \frac{27}{8n-3} + \frac{2}{8n-2} \right) \\
&= -\frac{49\pi}{12288\sqrt{2}} + \frac{25\pi}{8192} - \frac{31}{3072} \ln 2
\end{aligned}$$

$$\begin{aligned}
& + \frac{19}{1536\sqrt{2}} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} \frac{n^3}{(8n-5)(8n-4)(8n-3)(8n-2)(8n-1)} \\
& = \frac{1}{12288} \sum_{n=1}^{\infty} \left( \frac{125}{8n-5} - \frac{256}{8n-4} + \frac{162}{8n-3} - \frac{32}{8n-2} \right. \\
& \quad \left. + \frac{1}{8n-1} \right) \\
& = -\frac{19\pi}{49152\sqrt{2}} + \frac{17\pi}{49152} + \frac{17}{3072} \ln 2 \\
& \quad - \frac{143}{24576\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

由此顺次求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^3}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)} \\
& = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n^3}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-3)} \right. \\
& \quad \left. - \frac{n^3}{(8n-6)(8n-5)(8n-4)(8n-3)(8n-2)} \right] \\
& = \frac{243\pi}{81920\sqrt{2}} - \frac{461\pi}{245760} + \frac{37}{7680} \ln 2 \\
& \quad - \frac{521}{122880\sqrt{2}} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} \frac{n^3}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)} \\
& = -\frac{59\pi}{81920\sqrt{2}} + \frac{133\pi}{245760} - \frac{1}{320} \ln 2 \\
& \quad + \frac{149}{40960\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$



$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^3}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)} \\
&= \frac{1}{6} \sum_{n=1}^{\infty} \left[ \frac{n^3}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)} \right. \\
&\quad \left. - \frac{n^3}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)} \right] \\
&= \frac{151\pi}{245760\sqrt{2}} - \frac{33\pi}{81920} + \frac{61}{46080} \ln 2 \\
&\quad - \frac{121}{92160\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

仿此求得

$$\begin{aligned}
(一) \quad & \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)} \\
&= \frac{1}{144} \sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{9}{12n-9} + \frac{16}{12n-8} - \frac{9}{12n-7} \right. \\
&\quad \left. + \frac{1}{12n-5} \right) \\
&= \frac{49\pi}{3456\sqrt{3}} - \frac{\pi}{128} + \frac{1}{128} \ln 3 - \frac{1}{36} \ln 2 \\
&\quad + \frac{1}{64\sqrt{3}} \ln(2+\sqrt{3}), \\
&\sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)} \\
&= \frac{1}{144} \sum_{n=1}^{\infty} \left( \frac{1}{12n-5} - \frac{9}{12n-3} + \frac{16}{12n-2} - \frac{9}{12n-1} \right. \\
&\quad \left. + \frac{1}{12n+1} \right) \\
&= -\frac{5\pi}{1152\sqrt{3}} + \frac{\pi}{128} + \frac{1}{128} \ln 3 - \frac{1}{108} \ln 2
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{64\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{144}, \\
& \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)} \\
& = \frac{1}{360} \sum_{n=1}^{\infty} \left( \frac{5}{12n-11} - \frac{12}{12n-10} + \frac{20}{12n-8} - \frac{15}{12n-7} \right. \\
& \quad \left. + \frac{2}{12n-5} \right) \\
& = \frac{5\pi}{864\sqrt{3}} - \frac{\pi}{360} - \frac{1}{90} \ln 2 + \frac{1}{80\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)} \\
& = \frac{1}{360} \sum_{n=1}^{\infty} \left( \frac{5}{12n-5} - \frac{12}{12n-4} + \frac{20}{12n-2} - \frac{15}{12n-1} \right. \\
& \quad \left. + \frac{2}{12n+1} \right) \\
& = \frac{\pi}{480\sqrt{3}} + \frac{\pi}{360} + \frac{1}{270} \ln 2 - \frac{1}{80\sqrt{3}} \ln(2 + \sqrt{3}) \\
& \quad - \frac{1}{180}, \\
& \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)} \\
& = \frac{1}{240} \sum_{n=1}^{\infty} \left( \frac{5}{12n-11} - \frac{16}{12n-10} + \frac{15}{12n-9} - \frac{5}{12n-7} \right. \\
& \quad \left. + \frac{1}{12n-5} \right) \\
& = -\frac{\pi}{384\sqrt{3}} + \frac{13\pi}{5760} + \frac{1}{180} \ln 2 - \frac{1}{128} \ln 3 \\
& \quad + \frac{\sqrt{3}}{320} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)}$$

$$= \frac{1}{240} \sum_{n=1}^{\infty} \left( \frac{5}{12n-5} - \frac{16}{12n-4} + \frac{15}{12n-3} - \frac{5}{12n-1} + \frac{1}{12n+1} \right)$$

$$= \frac{49\pi}{5760\sqrt{3}} - \frac{13\pi}{5760} + \frac{1}{60} \ln 2 - \frac{1}{128} \ln 3 - \frac{\sqrt{3}}{320} \ln(2 + \sqrt{3}) - \frac{1}{240},$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)}$$

$$= \frac{1}{360} \sum_{n=1}^{\infty} \left( \frac{10}{12n-11} - \frac{36}{12n-10} + \frac{45}{12n-9} - \frac{20}{12n-8} + \frac{1}{12n-5} \right)$$

$$= -\frac{19\pi}{1728\sqrt{3}} + \frac{7\pi}{960} + \frac{1}{45} \ln 2 - \frac{1}{64} \ln 3 + \frac{1}{160\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)}$$

$$= \frac{1}{360} \sum_{n=1}^{\infty} \left( \frac{10}{12n-5} - \frac{36}{12n-4} + \frac{45}{12n-3} - \frac{20}{12n-2} + \frac{1}{12n+1} \right)$$

$$= \frac{43\pi}{2880\sqrt{3}} - \frac{7\pi}{960} + \frac{4}{135} \ln 2 - \frac{1}{64} \ln 3 - \frac{1}{160\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{360},$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)} \\
&= \frac{1}{120} \sum_{n=1}^{\infty} \left( \frac{4}{12n-10} - \frac{15}{12n-9} + \frac{20}{12n-8} - \frac{10}{12n-7} \right. \\
&\quad \left. + \frac{1}{12n-5} \right) \\
&= \frac{13\pi}{576\sqrt{3}} - \frac{37\pi}{2880} + \frac{1}{64} \ln 3 - \frac{2}{45} \ln 2 \\
&\quad + \frac{\sqrt{3}}{160} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)} \\
&= \frac{1}{120} \sum_{n=1}^{\infty} \left( \frac{4}{12n-4} - \frac{15}{12n-3} + \frac{20}{12n-2} - \frac{10}{12n-1} \right. \\
&\quad \left. + \frac{1}{12n+1} \right) \\
&= -\frac{31\pi}{2880\sqrt{3}} + \frac{37\pi}{2880} + \frac{1}{64} \ln 3 - \frac{1}{45} \ln 2 \\
&\quad - \frac{\sqrt{3}}{160} \ln(2 + \sqrt{3}) - \frac{1}{120}.
\end{aligned}$$

由此求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-3)(6n-2)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)} \right. \\
&\quad \left. - \frac{1}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)} \right] \\
&= \frac{\pi}{54\sqrt{3}} - \frac{\pi}{64} - \frac{1}{54} \ln 2 + \frac{1}{32\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{144},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-3)(6n-2)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)} \right. \\
&\quad \left. + \frac{1}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)} \right] \\
&= \frac{17\pi}{1728\sqrt{3}} + \frac{1}{64} \ln 3 - \frac{1}{27} \ln 2 - \frac{1}{144},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-4)(6n-2)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)} \right. \\
&\quad \left. - \frac{1}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)} \right] \\
&= \frac{\pi}{270\sqrt{3}} - \frac{\pi}{180} - \frac{2}{135} \ln 2 + \frac{1}{40\sqrt{3}} \ln(2+\sqrt{3}) \\
&\quad + \frac{1}{180},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-2)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)} \right. \\
&\quad \left. + \frac{1}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)} \right] \\
&= \frac{17\pi}{2160\sqrt{3}} - \frac{1}{135} \ln 2 - \frac{1}{180},
\end{aligned}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-1)(6n+1)}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)} \right. \\
&\quad \left. - \frac{1}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)} \right] \\
&= -\frac{\pi}{90\sqrt{3}} + \frac{13\pi}{2880} - \frac{1}{90} \ln 2 + \frac{\sqrt{3}}{160} \ln(2 + \sqrt{3}) \\
&\quad + \frac{1}{240},
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)} \right. \\
&\quad \left. + \frac{1}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)} \right] \\
&= \frac{17\pi}{2880\sqrt{3}} + \frac{1}{45} \ln 2 - \frac{1}{64} \ln 3 - \frac{1}{240},
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)} \right. \\
&\quad \left. - \frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)} \right] \\
&= -\frac{7\pi}{270\sqrt{3}} + \frac{7\pi}{480} - \frac{1}{135} \ln 2 + \frac{1}{80\sqrt{3}} \ln(2 + \sqrt{3}) \\
&\quad + \frac{1}{360},
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)} \Big] \\
& = \frac{17\pi}{4320\sqrt{3}} + \frac{7}{135} \ln 2 - \frac{1}{32} \ln 3 - \frac{1}{360}, \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)} \right. \\
& \quad \left. - \frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)} \right] \\
& = \frac{\pi}{30\sqrt{3}} - \frac{37\pi}{1440} - \frac{1}{45} \ln 2 + \frac{\sqrt{3}}{80} \ln(2+\sqrt{3}) + \frac{1}{120}, \\
& \sum_{n=1}^{\infty} \frac{1}{(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)} \right. \\
& \quad \left. + \frac{1}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)} \right] \\
& = \frac{17\pi}{1440\sqrt{3}} + \frac{1}{32} \ln 3 - \frac{1}{15} \ln 2 - \frac{1}{120}.
\end{aligned}$$

$$\begin{aligned}
(二) \quad & \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)} \\
& = \frac{1}{1728} \sum_{n=1}^{\infty} \left( \frac{11}{12n-11} - \frac{81}{12n-9} + \frac{128}{12n-8} - \frac{63}{12n-7} \right. \\
& \quad \left. + \frac{5}{12n-5} \right) \\
& = \frac{365\pi}{41472\sqrt{3}} - \frac{65\pi}{13824} + \frac{3}{512} \ln 3 - \frac{1}{54} \ln 2
\end{aligned}$$

$$\begin{aligned}
& + \frac{23}{2304\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)} \\
& = \frac{1}{1728} \sum_{n=1}^{\infty} \left( \frac{5}{12n-5} - \frac{27}{12n-3} + \frac{32}{12n-2} - \frac{9}{12n-1} \right. \\
& \quad \left. - \frac{1}{12n+1} \right) \\
& = -\frac{19\pi}{13824\sqrt{3}} + \frac{11\pi}{13824} + \frac{1}{512} \ln 3 - \frac{1}{648} \ln 2 \\
& \quad - \frac{5}{2304\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{1782}, \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)} \\
& = \frac{1}{864} \sum_{n=1}^{\infty} \left( \frac{11}{12n-11} - \frac{24}{12n-10} + \frac{32}{12n-8} - \frac{21}{12n-7} \right. \\
& \quad \left. + \frac{2}{12n-5} \right) \\
& = \frac{31\pi}{10368\sqrt{3}} - \frac{\pi}{864} - \frac{1}{144} \ln 2 \\
& \quad + \frac{5}{576\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)} \\
& = \frac{1}{4320} \sum_{n=1}^{\infty} \left( \frac{25}{12n-5} - \frac{48}{12n-4} + \frac{40}{12n-2} - \frac{15}{12n-1} \right. \\
& \quad \left. - \frac{2}{12n+1} \right) \\
& = \frac{7\pi}{17280\sqrt{3}} - \frac{\pi}{4320} + \frac{13}{6480} \ln 2
\end{aligned}$$



$$-\frac{7}{2880\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{2160},$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)}$$

$$= \frac{1}{576} \sum_{n=1}^{\infty} \left( \frac{11}{12n-11} - \frac{32}{12n-10} + \frac{27}{12n-9} - \frac{7}{12n-7} \right. \\ \left. + \frac{1}{12n-5} \right)$$

$$= -\frac{13\pi}{4608\sqrt{3}} + \frac{11\pi}{4608} + \frac{1}{216} \ln 2 - \frac{3}{512} \ln 3 \\ + \frac{17}{2304\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)}$$

$$= \frac{1}{2880} \sum_{n=1}^{\infty} \left( \frac{25}{12n-5} - \frac{64}{12n-4} + \frac{45}{12n-3} - \frac{5}{12n-1} \right. \\ \left. - \frac{1}{12n+1} \right)$$

$$= \frac{151\pi}{69120\sqrt{3}} - \frac{29\pi}{23040} + \frac{1}{180} \ln 2 - \frac{1}{512} \ln 3 \\ - \frac{31}{11520\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{2280},$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)}$$

$$= \frac{1}{864} \sum_{n=1}^{\infty} \left( \frac{22}{12n-11} - \frac{72}{12n-10} + \frac{81}{12n-9} - \frac{32}{12n-8} \right. \\ \left. + \frac{1}{12n-5} \right)$$

$$= -\frac{179\pi}{20736\sqrt{3}} + \frac{41\pi}{6912} + \frac{7}{432} \ln 2 - \frac{3}{256} \ln 3$$

$$+ \frac{7}{1152\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)}$$

$$= \frac{1}{4320} \sum_{n=1}^{\infty} \left( \frac{50}{12n-5} - \frac{144}{12n-4} + \frac{135}{12n-3} - \frac{40}{12n-2} \right.$$

$$\left. - \frac{1}{12n+1} \right)$$

$$= \frac{137\pi}{34560\sqrt{3}} - \frac{79\pi}{34560} + \frac{59}{6480} \ln 2 - \frac{1}{256} \ln 3$$

$$- \frac{17}{5760\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{4320},$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$= \frac{1}{288} \sum_{n=1}^{\infty} \left( \frac{8}{12n-10} - \frac{27}{12n-9} + \frac{32}{12n-8} - \frac{14}{12n-7} \right.$$

$$\left. + \frac{1}{12n-5} \right)$$

$$= \frac{101\pi}{6912\sqrt{3}} - \frac{19\pi}{2304} + \frac{3}{256} \ln 3 - \frac{13}{432} \ln 2$$

$$+ \frac{13}{1152\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)}$$

$$= \frac{1}{1440} \sum_{n=1}^{\infty} \left( \frac{16}{12n-4} - \frac{45}{12n-3} + \frac{40}{12n-2} - \frac{10}{12n-1} \right.$$

$$\left. - \frac{1}{12n+1} \right)$$

$$= -\frac{109\pi}{34560\sqrt{3}} + \frac{7\pi}{3840} + \frac{1}{256} \ln 3 - \frac{11}{2160} \ln 2$$

$$- \frac{11}{5760\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{1440}.$$

由此求得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)} \right.$$

$$\left. - \frac{2n}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)} \right]$$

$$= 2 \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$- \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)}$$

$$- 2 \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)}$$

$$= \frac{\pi}{162\sqrt{3}} - \frac{11\pi}{3456} - \frac{1}{162} \ln 2$$

$$+ \frac{5}{576\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{864},$$

$$\sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-9)(12n-8)(12n-7)(12n-5)} \right.$$

$$\left. + \frac{2n}{(12n-5)(12n-3)(12n-2)(12n-1)(12n+1)} \right]$$

$$\begin{aligned}
&= \frac{7\pi}{10368\sqrt{3}} + \frac{1}{128} \ln 3 - \frac{1}{81} \ln 2 + \frac{1}{864}, \\
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-4)(6n-2)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)} \right. \\
&\quad \left. - \frac{2n}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)} \right] \\
&= -\frac{\pi}{1620\sqrt{3}} + \frac{\pi}{1080} - \frac{11}{1620} \ln 2 \\
&\quad + \frac{7}{720\sqrt{3}} \ln(2+\sqrt{3}) - \frac{1}{1080}, \\
&\sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(6n-2)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)(12n-8)(12n-7)(12n-5)} \right. \\
&\quad \left. + \frac{2n}{(12n-5)(12n-4)(12n-2)(12n-1)(12n+1)} \right] \\
&= \frac{13\pi}{12960\sqrt{3}} + \frac{1}{810} \ln 2 + \frac{1}{1080}, \\
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-4)(6n-3)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)} \right. \\
&\quad \left. - \frac{2n}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)} \right] \\
&= -\frac{\pi}{135\sqrt{3}} + \frac{29\pi}{5760} - \frac{1}{135} \ln 2 \\
&\quad + \frac{31}{2880\sqrt{3}} \ln(2+\sqrt{3}) - \frac{1}{1440},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(6n-3)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)} \right. \\
&\quad \left. + \frac{2n}{(12n-5)(12n-4)(12n-3)(12n-1)(12n+1)} \right] \\
&= \frac{23\pi}{17280\sqrt{3}} + \frac{2}{135} \ln 2 - \frac{1}{128} \ln 3 + \frac{1}{1440},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)} \right. \\
&\quad \left. - \frac{2n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)} \right] \\
&= -\frac{23\pi}{1620\sqrt{3}} + \frac{79\pi}{8640} - \frac{13}{1620} \ln 2 \\
&\quad + \frac{17}{1440\sqrt{3}} \ln(2+\sqrt{3}) - \frac{1}{2160},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)} \right. \\
&\quad \left. + \frac{2n}{(12n-5)(12n-4)(12n-3)(12n-2)(12n+1)} \right] \\
&= \frac{43\pi}{25920\sqrt{3}} + \frac{23}{810} \ln 2 - \frac{1}{64} \ln 3 + \frac{1}{2160},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)} \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{2n}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)} \Big] \\
& = \frac{7\pi}{540\sqrt{3}} - \frac{7\pi}{960} - \frac{1}{180} \ln 2 + \frac{11}{1440\sqrt{3}} \ln(2 + \sqrt{3}) \\
& \quad - \frac{1}{720},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)} \right. \\
& \quad \left. + \frac{2n}{(12n-4)(12n-3)(12n-2)(12n-1)(12n+1)} \right] \\
& = \frac{\pi}{2880\sqrt{3}} + \frac{1}{64} \ln 3 - \frac{7}{270} \ln 2 + \frac{1}{720}.
\end{aligned}$$

$$\begin{aligned}
(\text{三}) \quad & \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-1)} \\
& = \frac{1}{360} \sum_{n=1}^{\infty} \left( \frac{10}{8n-7} - \frac{36}{8n-6} + \frac{45}{8n-5} - \frac{20}{8n-4} \right. \\
& \quad \left. + \frac{1}{8n-1} \right) \\
& = \frac{3\pi}{160\sqrt{2}} - \frac{\pi}{80} + \frac{19}{720} \ln 2 - \frac{17}{720\sqrt{2}} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-4)(8n-3)(8n-2)(8n-1)} \\
& = \frac{1}{360} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} - \frac{20}{8n-4} + \frac{45}{8n-3} - \frac{36}{8n-2} \right. \\
& \quad \left. + \frac{10}{8n-1} \right)
\end{aligned}$$

$$= -\frac{3\pi}{160\sqrt{2}} + \frac{\pi}{80} + \frac{19}{720} \ln 2$$

$$- \frac{17}{720\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-1)}$$

$$= \frac{1}{240} \sum_{n=1}^{\infty} \left( \frac{5}{8n-7} - \frac{16}{8n-6} + \frac{15}{8n-5} - \frac{5}{8n-3} \right. \\ \left. + \frac{1}{8n-1} \right)$$

$$= \frac{\pi}{80\sqrt{2}} - \frac{\pi}{120} + \frac{1}{120} \ln 2 - \frac{1}{240\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(8n-4)(8n-3)(8n-1)}$$

$$= \frac{1}{144} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} - \frac{9}{8n-5} + \frac{16}{8n-4} - \frac{9}{8n-3} \right. \\ \left. + \frac{1}{8n-1} \right)$$

$$= -\frac{1}{36} \ln 2 + \frac{5}{144\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(8n-3)(8n-2)(8n-1)}$$

$$= \frac{1}{240} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} - \frac{5}{8n-5} + \frac{15}{8n-3} - \frac{16}{8n-2} \right. \\ \left. + \frac{5}{8n+1} \right)$$

$$= -\frac{\pi}{80\sqrt{2}} + \frac{\pi}{120} + \frac{1}{120} \ln 2$$

$$- \frac{1}{240\sqrt{2}} \ln(\sqrt{2} + 1).$$

$$\begin{aligned}
(四) \quad & \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)(8n-4)(8n-1)} \\
&= \frac{1}{2880} \sum_{n=1}^{\infty} \left( \frac{70}{8n-7} - \frac{216}{8n-6} + \frac{225}{8n-5} - \frac{80}{8n-4} \right. \\
&\quad \left. + \frac{1}{8n-1} \right) \\
&= \frac{49\pi}{3840\sqrt{2}} - \frac{31\pi}{3840} + \frac{47}{2880} \ln 2 \\
&\quad - \frac{77}{5760\sqrt{2}} \ln(\sqrt{2}+1), \\
&\sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-4)(8n-3)(8n-2)(8n-1)} \\
&= \frac{1}{2880} \sum_{n=1}^{\infty} \left( \frac{7}{8n-7} - \frac{80}{8n-4} + \frac{135}{8n-3} - \frac{72}{8n-2} \right. \\
&\quad \left. + \frac{10}{8n-1} \right) \\
&= -\frac{23\pi}{3840\sqrt{2}} + \frac{17\pi}{3840} + \frac{29}{2880} \ln 2 \\
&\quad - \frac{59}{5760\sqrt{2}} \ln(\sqrt{2}+1), \\
&\sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-1)} \\
&= \frac{1}{1920} \sum_{n=1}^{\infty} \left( \frac{35}{8n-7} - \frac{96}{8n-6} + \frac{75}{8n-5} - \frac{15}{8n-3} \right. \\
&\quad \left. + \frac{1}{8n-1} \right) \\
&= \frac{31\pi}{3840\sqrt{2}} - \frac{19\pi}{3840} + \frac{1}{160} \ln 2 \\
&\quad - \frac{1}{320\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$



$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-5)(8n-4)(8n-3)(8n-1)} \\
&= \frac{1}{1152} \sum_{n=1}^{\infty} \left( \frac{7}{8n-7} - \frac{45}{8n-5} + \frac{64}{8n-4} - \frac{27}{8n-3} \right. \\
&\quad \left. + \frac{1}{8n-1} \right) \\
&= -\frac{\pi}{768\sqrt{2}} + \frac{\pi}{768} - \frac{1}{72} \ln 2 \\
&\quad + \frac{5}{288\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-5)(8n-3)(8n-2)(8n-1)} \\
&= \frac{1}{1920} \sum_{n=1}^{\infty} \left( \frac{7}{8n-7} - \frac{25}{8n-5} + \frac{45}{8n-3} - \frac{32}{8n-2} \right. \\
&\quad \left. + \frac{5}{8n-1} \right) \\
&= -\frac{17\pi}{3840\sqrt{2}} + \frac{13\pi}{3840} + \frac{1}{480} \ln 2 \\
&\quad - \frac{1}{960\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

$$\begin{aligned}
5. \quad \sum_{n=1}^{\infty} & \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{D}{k(n-1)+d} \right. \\
& \left. - \frac{E}{k(n-1)+e} + \frac{F}{k(n-1)+f} - \frac{A+D+F-B-E}{k(n-1)+g} \right]
\end{aligned}$$

仿照以上几个定理的证法，即可证明下面这个定理。

**定理13** 设  $A, B, D, E, F, b, d, e, f, g$  都是正整数，则

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(n-1)+b} + \frac{D}{k(n-1)+d} \right. \\
& \quad \left. - \frac{E}{k(n-1)+e} + \frac{F}{k(n-1)+f} - \frac{A+D+F-B-E}{k(n-1)+g} \right] \\
& = AC(a, k) - BC(b, k) + DC(d, k) - EC(e, k) \\
& \quad + FC(f, k) - (A+D+F-B-E)C(g, k) \\
& \quad + \frac{1}{k} \ln \frac{b^B e^E g^{A+D+F-B-E}}{a^A d^D f^F}.
\end{aligned}$$

例1 求和:

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-4)(6n-3)\cdots(6n+1)} \\
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}.
\end{aligned}$$

解 用分项分式方法求得

$$\begin{aligned}
& \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)} \\
& = \frac{1}{720} \left( \frac{5}{12n-11} - \frac{24}{12n-10} + \frac{45}{12n-9} - \frac{40}{12n-8} \right. \\
& \quad \left. + \frac{15}{12n-7} - \frac{1}{12n-5} \right).
\end{aligned}$$

于定理 13 令  $A=5$ ,  $B=24$ ,  $D=45$ ,  $E=40$ ,  $F=15$ ,  
 $a=1$ ,  $b=2$ ,  $d=3$ ,  $e=4$ ,  $f=5$ ,  $g=7$ ,  $k=12$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)} \\
& = \frac{1}{720} \sum_{n=1}^{\infty} \left( \frac{5}{12n-11} - \frac{24}{12n-10} + \frac{45}{12n-9} - \frac{40}{12n-8} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{15}{12n-7} - \frac{1}{12n-1} \Big) \\
& = \frac{1}{144} C(1,12) - \frac{1}{60} C(1,6) + \frac{1}{48} C(1,4) - \frac{1}{72} C(1,3) \\
& \quad + \frac{1}{48} C(5,12) - \frac{1}{720} C(7,12) + \frac{1}{8640} \ln \frac{2^{10 \cdot 7}}{3^{45} 5^{16}}.
\end{aligned}$$

根据定理 6, 有

$$\begin{aligned}
& \frac{1}{144} C(1,12) - \frac{1}{288} C(1,6) \\
& = \frac{\pi}{1728} + \frac{1}{576\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \frac{19}{2160} C(1,4) - \frac{19}{1440} C(1,6) \\
& = -\frac{19\pi}{5760\sqrt{3}} + \frac{19\pi}{17280} - \frac{1}{17280} \ln 3^{10}, \\
& \frac{13}{1080} C(1,4) - \frac{13}{1440} C(1,3) \\
& = -\frac{13\pi}{8640\sqrt{3}} + \frac{13\pi}{8640} + \frac{1}{8640} \ln \frac{2^{20}}{3^{13}}, \\
& \frac{7}{360} C(5,12) - \frac{7}{1440} C(1,3) \\
& = -\frac{7\pi}{2160\sqrt{3}} + \frac{7\pi}{4320} + \frac{1}{4320} \ln 2^7 \cdot 5^7 \\
& \quad - \frac{7}{1440\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \frac{1}{720} C(5,12) - \frac{1}{720} C(7,12) \\
& = -\frac{\pi}{2880\sqrt{3}} + \frac{\pi}{4320} + \frac{1}{8640} \ln \frac{5}{7}.
\end{aligned}$$

于是

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-5)} \\ = -\frac{29\pi}{3456\sqrt{3}} + \frac{29\pi}{5760} + \frac{1}{60} \ln 2 - \frac{1}{128} \ln 3 \\ - \frac{1}{320\sqrt{3}} \ln(2 + \sqrt{3}). \end{aligned}$$

仿此求得

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(12n-5)(12n-4)(12n-3)(12n-2)(12-1)(12+1)} \\ = \frac{1}{720} \sum_{n=1}^{\infty} \left( \frac{5}{12n-5} - \frac{24}{12n-4} + \frac{45}{12n-3} - \frac{40}{12n-2} \right. \\ \left. + \frac{15}{12n-1} - \frac{1}{12n+1} \right) \\ = \frac{37\pi}{5760\sqrt{3}} - \frac{29\pi}{5760} + \frac{7}{540} \ln 2 - \frac{1}{128} \ln 3 \\ + \frac{1}{320\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{720}. \end{aligned}$$

$$\begin{aligned} \text{于是 } \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(6n-5)(6n-4)(6n-3)\cdots(6n+1)} \\ = \sum_{n=1}^{\infty} \left[ \frac{1}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)} \right. \\ \left. - \frac{1}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)} \right] \\ = -\frac{2\pi}{135\sqrt{3}} + \frac{29\pi}{2880} + \frac{1}{270} \ln 2 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{160\sqrt{3}} \ln(2+\sqrt{3}) - \frac{1}{720}, \\
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{2(1n-11)(12n-10)(12n-9)\cdots(12n-5)} \right. \\
& \quad \left. + \frac{1}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)} \right] \\
& = -\frac{17\pi}{8640\sqrt{3}} + \frac{4}{135} \ln 2 - \frac{1}{64} \ln 3 + \frac{1}{720}.
\end{aligned}$$

**例2** 求和:

$$\sum_{n=1}^{\infty} \frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)}.$$

**解** 用分项分式方法求得

$$\begin{aligned}
& \frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)} \\
& = \frac{1}{491520} \left( \frac{2401}{8n-7} - \frac{6480}{8n-6} + \frac{6250}{8n-5} - \frac{2560}{8n-4} \right. \\
& \quad \left. + \frac{405}{8n-3} - \frac{16}{8n-2} \right).
\end{aligned}$$

于定理 13 令  $A=2401$ ,  $B=6480$ ,  $D=6250$ ,  $E=2560$ ,  
 $F=405$ ,  $a=1$ ,  $b=2$ ,  $d=3$ ,  $e=4$ ,  $f=5$ ,  $g=6$ ,  $k=8$ ,  
 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)} \\
& = \frac{1}{491520} \sum_{n=1}^{\infty} \left( \frac{2401}{8n-7} - \frac{6480}{8n-6} + \frac{6250}{8n-5} - \frac{2560}{8n-4} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{405}{8n-3} - \frac{16}{8n-2} \Big) \\
& = \frac{2401}{491520} C(1,8) - \frac{27}{4096} C(1,4) + \frac{625}{49152} C(3,8) \\
& \quad - \frac{1}{768} C(1,2) + \frac{27}{32768} C(5,8) - \frac{1}{61440} C(3,4) \\
& \quad + \frac{1}{3932160} \ln \frac{2^{11616}}{3^{6234} 5^{405}}.
\end{aligned}$$

根据定理 6, 有

$$\begin{aligned}
& \frac{2401}{491520} C(1,8) - \frac{2401}{983040} C(1,4) \\
& = \frac{2401\pi}{3932160\sqrt{2}} + \frac{2401}{1966080\sqrt{2}} \ln(\sqrt{2}+1), \\
& \frac{4079}{491520} C(3,8) - \frac{4079}{983040} C(1,4) \\
& = \frac{4079\pi}{3932160\sqrt{2}} - \frac{4079\pi}{3932160} + \frac{1}{3932160} \ln 3^{4079} \\
& \quad - \frac{4079}{1966080\sqrt{2}} - \ln(\sqrt{2}+1), \\
& \frac{2171}{491520} C(3,8) - \frac{2171}{1966080} C(1,2) \\
& = \frac{2171\pi}{3932160\sqrt{2}} - \frac{2171\pi}{7864320} + \frac{1}{3932160} \ln 3^{2171} \\
& \quad - \frac{2171}{1966080\sqrt{2}} \ln(\sqrt{2}+1), \\
& \frac{389}{491520} C(5,8) - \frac{389}{1966080} C(1,2) \\
& = -\frac{389\pi}{3932160\sqrt{2}} + \frac{389\pi}{7864320} + \frac{1}{3932160} \ln 5^{389}
\end{aligned}$$

$$\begin{aligned}
& -\frac{389}{1966080\sqrt{2}} \ln(\sqrt{2}+1), \\
& \frac{1}{30720} C(5,8) - \frac{1}{61440} C(3,4) \\
& = -\frac{\pi}{245760\sqrt{2}} + \frac{\pi}{245760} + \frac{1}{245760} \ln \frac{5}{3} \\
& \quad - \frac{1}{122880\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

于是

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-2)} \\
& = \frac{4123\pi}{1966080\sqrt{2}} - \frac{2477\pi}{1966080} + \frac{121}{40960} \ln 2 \\
& \quad - \frac{709}{327680\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

仿此求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^4}{(8n-6)(8n-5)(8n-4)\cdots(8n-1)} \\
& = \frac{1}{491520} \sum_{n=1}^{\infty} \left( \frac{1296}{8n-6} - \frac{3125}{8n-5} + \frac{2560}{8n-4} - \frac{810}{8n-3} \right. \\
& \quad \left. + \frac{80}{8n-2} - \frac{1}{8n-1} \right) \\
& = -\frac{1157\pi}{1966080\sqrt{2}} + \frac{883\pi}{1966080} - \frac{203}{122880} \ln 2 \\
& \quad + \frac{1967}{983040\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

于是

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^4}{(8n-7)(8n-6)(8n-5)\cdots(8n-1)} \\
& = \frac{11\pi}{24576\sqrt{2}} - \frac{7\pi}{24576} + \frac{283}{368640} \ln 2
\end{aligned}$$

$$= \frac{2047}{2949120\sqrt{2}} \ln(\sqrt{2} + 1).$$

仿此求得

$$\begin{aligned} \text{(一)} \quad & \sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)\cdots(12n-5)} \\ &= \frac{1}{1728} \sum_{n=1}^{\infty} \left( \frac{11}{12n-11} - \frac{48}{12n-10} + \frac{81}{12n-9} - \frac{64}{12n-8} \right. \\ &\quad \left. + \frac{21}{12n-7} - \frac{1}{12n-5} \right) \\ &= -\frac{241\pi}{41472\sqrt{3}} + \frac{49\pi}{13824} + \frac{5}{432} \ln 2 \\ &\quad - \frac{3}{512} \ln 3 - \frac{1}{768\sqrt{3}} \ln(2 + \sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(12n-5)(12n-4)(12n-3)(12n-2)\cdots(12n+1)} \\ &= \frac{1}{8640} \sum_{n=1}^{\infty} \left( \frac{25}{12n-5} - \frac{96}{12n-4} + \frac{135}{12n-3} - \frac{80}{12n-2} \right. \\ &\quad \left. + \frac{15}{12n-1} + \frac{1}{12n+1} \right) \\ &= \frac{41\pi}{23040\sqrt{3}} - \frac{71\pi}{69120} + \frac{23}{6480} \ln 2 - \frac{1}{512} \ln 3 \\ &\quad - \frac{1}{3840\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{8640}. \end{aligned}$$

由此求得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(6n-5)(6n-4)(6n-3)\cdots(6n+1)}$$



$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)} \right. \\ \left. - \frac{2n}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)} \right]$$

$$= -\frac{11\pi}{1620\sqrt{3}} + \frac{71\pi}{17280} - \frac{1}{1620} \ln 2 \\ + \frac{1}{960\sqrt{3}} \ln(2+\sqrt{3}) + \frac{1}{4320},$$

$$\sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)} \\ = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)} \right. \\ \left. + \frac{2n}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)} \right]$$

$$= \frac{17\pi}{51840\sqrt{3}} + \frac{11}{810} \ln 2 - \frac{1}{128} \ln 3 - \frac{1}{4320}.$$

$$(二) \quad \sum_{n=1}^{\infty} \frac{n^2}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)}$$

$$= \frac{1}{20736} \sum_{n=1}^{\infty} \left( \frac{121}{12n-11} - \frac{480}{12n-10} + \frac{729}{12n-9} - \frac{512}{12n-8} \right. \\ \left. + \frac{147}{12n-7} - \frac{5}{12n-5} \right)$$

$$= -\frac{2045\pi}{497664\sqrt{3}} + \frac{425\pi}{165888} + \frac{7}{864} \ln 2 - \frac{27}{6144} \ln 3$$

$$-\frac{7}{27648\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)} \\ &= \frac{1}{103680} \sum_{n=1}^{\infty} \left( \frac{125}{12n-5} - \frac{384}{12n-4} + \frac{405}{12n-3} - \frac{160}{12n-2} \right. \\ & \quad \left. + \frac{15}{12n-1} + \frac{1}{12n+1} \right) \\ &= \frac{397\pi}{829440\sqrt{3}} - \frac{229\pi}{829440} + \frac{41}{38880} \ln 2 - \frac{1}{2048} \ln 3 \\ & \quad - \frac{37}{138240\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{103680}. \end{aligned}$$

由此求得

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(6n-5)(6n-4)(6n-3)\cdots(6n+1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)} \right. \\ & \quad \left. - \frac{4n^2}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)} \right] \\ &= -\frac{17\pi}{4860\sqrt{3}} + \frac{229\pi}{103680} - \frac{7}{4860} \ln 2 \\ & \quad + \frac{37}{17280\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{25920}, \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(6n-5)(6n-4)(6n-3)(6n-2)(6n-1)(6n+1)}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(12n-11)(12n-10)\cdots(12n-7)(12n-5)} \right. \\
&\quad \left. + \frac{4n^2}{(12n-5)(12n-4)(12n-3)\cdots(12n+1)} \right] \\
&= \frac{103\pi}{311040\sqrt{3}} + \frac{17}{2430} \ln 2 - \frac{1}{256} \ln 3 + \frac{1}{25920}.
\end{aligned}$$

(三)

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-7)(12n-5)(12n-1)(12n+1)} \\
&= \frac{1}{190080} \sum_{n=1}^{\infty} \left( \frac{66}{12n-11} - \frac{128}{12n-10} + \frac{165}{12n-7} - \frac{132}{12n-5} \right. \\
&\quad \left. + \frac{44}{12n-1} - \frac{15}{12n+1} \right) \\
&= -\frac{19\pi}{69120\sqrt{3}} + \frac{19\pi}{142560} + \frac{1}{17820} \ln 2 \\
&\quad + \frac{31}{380160\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{1}{12672}, \\
&\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-7)(12n-5)(12n-1)(12n+1)} \\
&= \frac{1}{2280960} \sum_{n=1}^{\infty} \left( \frac{726}{12n-11} - \frac{1280}{12n-10} + \frac{1155}{12n-7} \right. \\
&\quad \left. - \frac{660}{12n-5} + \frac{44}{12n-1} + \frac{15}{12n+1} \right) \\
&= -\frac{109\pi}{829440\sqrt{3}} + \frac{157\pi}{1710720} + \frac{1}{21384} \ln 2
\end{aligned}$$

$$+ \frac{29}{912384\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{152064},$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)(12n-1)}$$

$$= \frac{1}{17280} \sum_{n=1}^{\infty} \left( \frac{36}{12n-11} - \frac{128}{12n-10} + \frac{135}{12n-9} - \frac{60}{12n-7} \right. \\ \left. + \frac{18}{12n-5} - \frac{1}{12n-1} \right)$$

$$= -\frac{13\pi}{138240\sqrt{3}} + \frac{53\pi}{414720} + \frac{1}{1620} \ln 2 - \frac{1}{1024} \ln 3$$

$$+ \frac{77}{69120\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-7)(12n-5)(12n-1)}$$

$$= \frac{1}{207360} \sum_{n=1}^{\infty} \left( \frac{396}{12n-11} - \frac{1280}{12n-10} + \frac{1215}{12n-9} - \frac{420}{12n-7} \right. \\ \left. + \frac{90}{12n-5} - \frac{1}{12n-1} \right)$$

$$= -\frac{373\pi}{1658880\sqrt{3}} + \frac{989\pi}{4976640} + \frac{1}{1944} \ln 2 - \frac{3}{4096} \ln 3$$

$$+ \frac{145}{165888\sqrt{3}} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)(12n-1)}$$

$$= \frac{1}{20160} \sum_{n=1}^{\infty} \left( \frac{56}{12n-11} - \frac{224}{12n-10} + \frac{315}{12n-9} - \frac{160}{12n-8} \right.$$

$$\begin{aligned}
& + \frac{14}{12n-5} - \frac{1}{12n-1} \Big) \\
& = -\frac{619\pi}{483840\sqrt{3}} + \frac{401\pi}{483840} + \frac{11}{3780} \ln 2 - \frac{1}{512} \ln 3 \\
& \quad + \frac{41}{80640\sqrt{3}} \ln(2 + \sqrt{3}), \\
\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-5)(12n-1)} \\
& = \frac{1}{241920} \sum_{n=1}^{\infty} \left( \frac{616}{12n-11} - \frac{2240}{12n-10} + \frac{2835}{12n-9} - \frac{1280}{12n-8} \right. \\
& \quad \left. + \frac{70}{12n-5} - \frac{1}{12n-1} \right) \\
& = -\frac{5939\pi}{5806080\sqrt{3}} + \frac{3929\pi}{5806080} + \frac{19}{9072} \ln 2 - \frac{3}{2048} \ln 3 \\
& \quad + \frac{109}{193536\sqrt{3}} \ln(2 + \sqrt{3}), \\
\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)} \\
& = \frac{1}{30240} \sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{210}{12n-5} + \frac{720}{12n-4} - \frac{945}{12n-3} \right. \\
& \quad \left. + \frac{560}{12n-2} - \frac{126}{12n-1} \right) \\
& = -\frac{883\pi}{241920\sqrt{3}} + \frac{1619\pi}{725760} + \frac{1}{256} \ln 3 - \frac{17}{2268} \ln 2 \\
& \quad + \frac{17}{24192\sqrt{3}} \ln(2 + \sqrt{3}), \\
\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-5)(12n-4)(12n-3)(12n-2)(12n-1)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{362880} \sum_{n=1}^{\infty} \left( \frac{11}{12n-11} - \frac{1050}{12n-5} + \frac{2880}{12n-4} - \frac{2835}{12n-3} \right. \\
&\quad \left. + \frac{1120}{12n-2} - \frac{126}{12n-1} \right) \\
&= -\frac{2993\pi}{2903040\sqrt{3}} + \frac{5209\pi}{8709120} + \frac{1}{1024} \ln 3 - \frac{61}{27216} \ln 2 \\
&\quad + \frac{187}{290304\sqrt{3}} \ln(2 + \sqrt{3}), \\
\sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-1)} \\
&= \frac{1}{30240} \sum_{n=1}^{\infty} \left( \frac{126}{12n-11} - \frac{560}{12n-10} + \frac{945}{12n-9} - \frac{720}{12n-8} \right. \\
&\quad \left. + \frac{210}{12n-7} - \frac{1}{12n-1} \right) \\
&= -\frac{883\pi}{241920\sqrt{3}} + \frac{1619\pi}{725760} + \frac{17}{2268} \ln 2 - \frac{1}{256} \ln 3 \\
&\quad - \frac{17}{24192\sqrt{3}} \ln(2 + \sqrt{3}), \\
\sum_{n=1}^{\infty} \frac{n}{(12n-11)(12n-10)(12n-9)(12n-8)(12n-7)(12n-1)} \\
&= \frac{1}{362880} \sum_{n=1}^{\infty} \left( \frac{1386}{12n-11} - \frac{5600}{12n-10} + \frac{8505}{12n-9} - \frac{5760}{12n-8} \right. \\
&\quad \left. + \frac{1470}{12n-7} - \frac{1}{12n-1} \right) \\
&= -\frac{7603\pi}{2903040\sqrt{3}} + \frac{14219\pi}{8709120} + \frac{143}{27216} \ln 2 - \frac{3}{1024} \ln 3 \\
&\quad - \frac{17}{290304\sqrt{3}} \ln(2 + \sqrt{3}).
\end{aligned}$$

$$\begin{aligned}
(四) \quad & \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-1)(8n+1)} \\
&= \frac{1}{13440} \sum_{n=1}^{\infty} \left( \frac{35}{8n-7} - \frac{128}{8n-6} + \frac{140}{8n-5} - \frac{70}{8n-3} \right. \\
&\quad \left. + \frac{28}{8n-1} - \frac{5}{8n+1} \right) \\
&= \frac{53\pi}{26880\sqrt{2}} - \frac{\pi}{640} + \frac{1}{840} \ln 2 \\
&\quad - \frac{1}{4480\sqrt{2}} \ln(\sqrt{2}+1) + \frac{1}{2688}, \\
&\sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-1)(8n+1)} \\
&= \frac{1}{107520} \sum_{n=1}^{\infty} \left( \frac{245}{8n-7} - \frac{768}{8n-6} + \frac{700}{8n-5} - \frac{210}{8n-3} \right. \\
&\quad \left. + \frac{28}{8n-1} + \frac{5}{8n+1} \right) \\
&= \frac{283\pi}{215040\sqrt{2}} - \frac{13\pi}{15360} + \frac{1}{1120} \ln 2 \\
&\quad - \frac{53}{107520\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{21504}, \\
&\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(8n-4)(8n-3)(8n-1)(8n+1)} \\
&= \frac{1}{5760} \sum_{n=1}^{\infty} \left( \frac{5}{8n-7} - \frac{60}{8n-5} + \frac{128}{8n-4} - \frac{90}{8n-3} \right. \\
&\quad \left. + \frac{20}{8n-1} - \frac{3}{8n+1} \right) \\
&= \frac{\pi}{3840\sqrt{2}} - \frac{\pi}{1920} - \frac{1}{180} \ln 2 + \frac{43}{5760\sqrt{2}} \ln(\sqrt{2}+1) \\
&\quad + \frac{1}{1920},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-5)(8n-4)(8n-3)(8n-1)(8n+1)} \\
&= \frac{1}{46080} \sum_{n=1}^{\infty} \left( \frac{35}{8n-7} - \frac{300}{8n-5} + \frac{512}{8n-4} - \frac{270}{8n-3} \right. \\
&\quad \left. + \frac{20}{8n-1} + \frac{3}{8n+1} \right) \\
&= -\frac{\pi}{30720\sqrt{2}} + \frac{\pi}{15360} - \frac{1}{360} \ln 2 \\
&\quad + \frac{157}{46080\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{15360},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-2)(8n-1)} \\
&= \frac{1}{240} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} - \frac{4}{8n-6} + \frac{5}{8n-5} - \frac{5}{8n-3} \right. \\
&\quad \left. + \frac{4}{8n-2} - \frac{1}{8n-1} \right) \\
&= \frac{\pi}{160\sqrt{2}} - \frac{\pi}{240},
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-6)(8n-5)(8n-3)(8n-2)(8n-1)} \\
&= \frac{1}{1920} \sum_{n=1}^{\infty} \left( \frac{7}{8n-7} - \frac{24}{8n-6} + \frac{25}{8n-5} - \frac{15}{8n-3} + \frac{8}{8n-2} \right. \\
&\quad \left. - \frac{1}{8n-1} \right) \\
&= \frac{\pi}{320\sqrt{2}} - \frac{\pi}{480} + \frac{1}{960} \ln 2 \\
&\quad - \frac{1}{1920\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

在三、中已经看到, 当  $k=1, 2, 3, 4, 6$  时, 容易求得



$A(a, k)$  的有限表达式. 当  $k = 5$  或  $8$  时, 也可以求得  $A(a, k)$  的有限表达式, 但计算过程很复杂. 对于任意的  $A(a, k)$ , 难以求得其有限的表达式. 应用上述的定理可推得收敛迅速的级数来表示  $A(a, k)$ , 只要计算所得级数的前若干项之和, 即得到  $A(a, k)$  的近似值, 并达到很高的精确度.

例3 求  $A(1, 18)$ .

解 用分项分式方法求得

$$\begin{aligned} & \frac{1}{(36n-35)\cdots(36n-33)(36n-30)(36n-27)(36n-24)} \\ &= \frac{1}{498960} \left( \frac{567}{36n-35} - \frac{1782}{36n-34} + \frac{1540}{36n-33} - \frac{462}{36n-30} \right. \\ & \quad \left. + \frac{165}{36n-27} - \frac{28}{36n-24} \right). \end{aligned}$$

于定理13令  $A = 567$ ,  $B = 1782$ ,  $D = 1540$ ,  $E = 462$ ,  $F = 165$ ,  $a = 1$ ,  $b = 2$ ,  $d = 3$ ,  $e = 6$ ,  $f = 9$ ,  $g = 12$ ,  $k = 36$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(36n-35)\cdots(36n-33)(36n-30)(12n-27)(36n-24)} \\ &= \frac{1}{498960} \sum_{n=1}^{\infty} \left( \frac{567}{36n-35} - \frac{1782}{36n-34} + \frac{1540}{36n-33} \right. \\ & \quad \left. - \frac{462}{36n-30} + \frac{165}{36n-27} - \frac{28}{36n-24} \right) \\ &= \frac{1}{880} C(1, 36) - \frac{1}{560} C(1, 18) + \frac{1}{972} C(1, 12) \\ & \quad - \frac{1}{6480} C(1, 6) + \frac{1}{27216} C(1, 4) - \frac{1}{213840} C(1, 3) \\ & \quad + \frac{1}{17962560} \ln \frac{2^{2800}}{3^{1880}}. \end{aligned}$$

根据定理 5 与 6, 有

$$\begin{aligned}
 & \frac{1}{160380} C(1,4) - \frac{1}{213840} C(1,3) \\
 &= -\frac{\pi}{1283040\sqrt{3}} + \frac{\pi}{1283040} + \frac{1}{1283040} \ln \frac{2^3}{3}, \\
 & \frac{137}{4490640} C(1,4) - \frac{137}{2993760} C(1,6) \\
 &= -\frac{137\pi}{11975040\sqrt{3}} + \frac{137\pi}{35925120} \\
 & \quad - \frac{1}{35925120} \ln 3^{137}, \\
 & \frac{1}{972} C(1,12) - \frac{65}{598952} C(1,6) \\
 &= \frac{1}{1232} C(1,12) + \frac{65}{299376} C(1,12) - \frac{65}{598952} C(1,6) \\
 &= \frac{1}{1232} C(1,12) + \frac{65\pi}{3592512} \\
 & \quad + \frac{65}{1197504\sqrt{3}} \ln(2 + \sqrt{3}), \\
 & \frac{1}{880} C(1,36) - \frac{1}{560} C(1,18) \\
 &= \frac{1}{880} C(1,36) - \frac{1}{1760} C(1,18) - \frac{3}{2464} C(1,18) \\
 &= \frac{1}{1760} A(1,18) - \frac{3}{2464} C(1,18).
 \end{aligned}$$

于是

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{1}{(36n-35)\cdots(36n-33)(36n-30)(36n-27)(36n-24)} \\
 &= \frac{1}{1760} A(1,18) - \frac{3}{2464} C(1,18) + \frac{1}{1232} C(1,12)
 \end{aligned}$$

$$+ \frac{163\pi}{7185024} - \frac{439\pi}{35925120\sqrt{3}} + \frac{97}{748440} \ln 2$$

$$- \frac{65}{798336} \ln 3 + \frac{65}{1197504\sqrt{3}} \ln(2 + \sqrt{3}).$$

仿此推得

$$\sum_{n=1}^{\infty} \frac{1}{(36n-34)(36n-33)(36n-30)(36n-27)\cdots(36n-21)}$$

$$= \frac{1}{884520} \sum_{n=1}^{\infty} \left( \frac{243}{36n-34} - \frac{455}{36n-33} + \frac{455}{36n-30} \right.$$

$$\left. - \frac{390}{36n-27} + \frac{182}{36n-24} - \frac{35}{36n-21} \right)$$

$$= \frac{1}{7280} C(1,18) - \frac{1}{10920} C(1,12) - \frac{221\pi}{15921360}$$

$$+ \frac{127\pi}{7960680\sqrt{3}} + \frac{23}{884520} \ln 3$$

$$- \frac{311}{7960680} \ln 2 - \frac{59}{3538080\sqrt{3}} \ln(2 + \sqrt{3}).$$

由此解出  $A(1,18)$  即得

$$A(1,18)$$

$$= \sum_{n=1}^{\infty} \frac{1760}{(36n-35)\cdots(36n-33)(36n-30)(36n-27)(36n-24)}$$

$$+ \sum_{n=1}^{\infty} \frac{15600}{(36n-34)(36n-33)(36n-30)(36n-27)\cdots(36n-21)}$$

$$- \frac{1547\pi}{6804\sqrt{3}} + \frac{3605\pi}{20412} + \frac{1946}{5103} \ln 2 - \frac{595}{2268} \ln 3$$

$$+ \frac{560}{3402\sqrt{3}} \ln(2 + \sqrt{3}).$$

计算右端两级数之和的前10位小数，得

$$\sum_{n=1}^6 \frac{1760}{(36n-35)(36n-34)(36n-33)(36n-30)\cdots(36n-24)} \\ = 0.4526752601\cdots,$$

$$\sum_{n=1}^7 \frac{15600}{(36n-34)(36n-33)(36n-30)(36n-27)\cdots(36n-21)} \\ = 0.2674920561\cdots.$$

即得  $A(1,18) = 0.9638842503\cdots$ .

直接计算  $A(1,18) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{18n-17}$ ，很难达到

这样高的精确度。例如：

$$\sum_{n=1}^{370} (-1)^{n-1} \frac{1}{18n-17} = 0.9638001223\cdots,$$

这里只准确到小数点后四位数。

仿此求得

$A(1,8)$

$$= \sum_{n=1}^{\infty} \frac{270270}{(16n-15)(16n-14)(16n-12)\cdots(16n-2)} \\ + \frac{719\pi}{1024\sqrt{2}} - \frac{207\pi}{512} + \frac{5945}{2048} \ln 2 \\ - \frac{2951}{1024\sqrt{2}} \ln(\sqrt{2}+1).$$

而 
$$\sum_{n=1}^6 \frac{270270}{(16n-15)(16n-14)(16n-12)\cdots(16-2)} \\ = 0.4189491976\cdots,$$

所以  $A(1,8) = 0.9246517057\cdots$ ,

$$A(1,9)$$

$$= \sum_{n=1}^{\infty} \frac{1760}{(18n-17)(18n-16)(18n-15)(18n-12)\cdots(18n-6)} \\ + \sum_{n=1}^{\infty} \frac{15600}{(18n-16)(18n-15)(18n-12)(18n-9)\cdots(18n-3)} \\ + \frac{107\pi}{1458\sqrt{3}} + \frac{689}{729} \ln 2 - \frac{85}{162} \ln 3,$$

$$\text{而 } \sum_{n=1}^{10} \frac{1760}{(18n-17)(18n-16)(18n-15)(18n-12)\cdots(18n-6)} \\ = 0.4526866482\cdots,$$

$$\sum_{n=1}^{13} \frac{15600}{(18n-16)(18n-15)(18n-12)(18n-9)\cdots(18n-3)} \\ = 0.2675502583\cdots,$$

$$\text{所以 } A(1,9) = 0.9320304221\cdots,$$

$$A(1,12)$$

$$= \sum_{n=1}^{\infty} \frac{420}{(24n-23)(24n-22)\cdots(24n-20)(24n-18)(24n-16)} \\ + \frac{187\pi}{192\sqrt{3}} - \frac{7\pi}{12\sqrt{2}} - \frac{3\pi}{64} - \frac{17}{16} \ln 2 \\ + \frac{63}{64} \ln 3 + \frac{9\sqrt{3}}{32} \ln(2+\sqrt{3}) \\ - \frac{7}{6\sqrt{2}} \ln(\sqrt{2}+1),$$

而

$$\sum_{n=1}^6 \frac{420}{(24n-23)(24n-22)\cdots(24n-20)(24n-18)(24n-16)} \\ = 0.3645842477\cdots,$$

$$\text{所以 } A(1,12) = 0.9474696544\cdots,$$

$$A(1,15)$$

$$= \sum_{n=1}^{\infty} \frac{19152}{(30n-29)(30n-28)(30n-25)(30n-20)\cdots(30n-10)} \\ + \sum_{n=1}^{\infty} \frac{182160}{(30n-28)(30n-25)(30n-20)(30n-15)\cdots(30n-5)} \\ + \frac{56\pi}{1875\sqrt{3}} + \frac{829}{1875} \ln 2 - \frac{162}{625} \ln 3,$$

$$\text{而 } \sum_{n=1}^9 \frac{19152}{(30n-29)(30n-28)(30n-25)(30n-20)\cdots(30n-10)} \\ = 0.6384063414\cdots,$$

$$\sum_{n=1}^{12} \frac{182160}{(30n-28)(30n-25)(30n-20)(30n-15)\cdots(30n-5)} \\ = 0.2429143299\cdots,$$

$$\text{所以 } A(1,15) = 0.9571959804\cdots,$$

$$A(1,16)$$

$$= \sum_{n=1}^{\infty} \frac{6930}{(32n-31)(32n-30)(32n-28)(32n-24)\cdots(32n-16)} \\ + \frac{17}{32} \sum_{n=1}^{\infty} \frac{1890}{(16n-15)(16n-14)(16n-12)(16n-10)\cdots(16n-6)} \\ + \frac{3801\pi}{16384\sqrt{2}} - \frac{1109\pi}{8192} + \frac{2429}{8192} \ln 2 \\ - \frac{2121}{8192\sqrt{2}} \ln(\sqrt{2}+1),$$

而

$$\sum_{n=1}^7 \frac{6930}{(32n-31)(32n-30)(32n-28)(32n-24)\cdots(32n-16)} \\ = 0.5639669362\cdots,$$

$$\sum_{n=1}^{10} \frac{1890}{(16n-15)(16n-14)(16n-12)(16n-10)\cdots(16n-6)}$$

$$= 0.4922108387\cdots,$$

所以  $A(1,16) = 0.9596845382\cdots$ ;

$A(1,24)$

$$= \sum_{n=1}^{\infty} \frac{43010}{(48n-47)(48n-46)(48n-42)(48n-36)\cdots(48n-24)}$$

$$+ \sum_{n=1}^{\infty} \frac{415800}{(48n-46)(48n-42)(48n-36)(48n-30)\cdots(48n-18)}$$

$$+ \frac{15169\pi}{124416\sqrt{2}} - \frac{2245\pi}{31104} + \frac{9295}{62208} \ln 2$$

$$- \frac{8921}{62208\sqrt{2}} \ln(\sqrt{2}+1),$$

而

$$\sum_{n=1}^{\infty} \frac{43010}{(48n-47)(48n-46)(48n-42)(48n-36)\cdots(48n-24)}$$

$$= 0.6913913490\cdots,$$

$$\sum_{n=1}^{\infty} \frac{415800}{(48n-46)(48n-42)(48n-36)(48n-30)\cdots(48n-18)}$$

$$= 0.2228081072\cdots,$$

所以  $A(1,24) = 0.9724842827\cdots$ .

$$6. \quad \sum_{n=1}^{\infty} \left[ \frac{A}{k(An-A)+a} - \frac{B}{k(Bn-B)+b} \right]$$

**定理14** 设  $A$ 、 $B$ 、 $b$  都是正整数, 则

$$\sum_{n=1}^{\infty} \left[ \frac{A}{k(An-A)+a} - \frac{B}{k(Bn-B)+b} \right] \\ = AC(a, Ak) - BC(b, Bk) + \frac{1}{k} \ln \frac{Ab}{Ba}.$$

**证明** 根据定理 2, 有

$$\sum_{i=1}^n \left[ \frac{A}{k(Ai-A)+a} - \frac{B}{k(Bi-B)+b} \right] \\ = AC(a, Ak) - BC(b, Bk) + \frac{1}{k} \ln \frac{Ank - Ak + a}{Bnk - Bk + b} \\ + \frac{1}{k} \ln \frac{b}{a} + A\varepsilon_n(a, Ak) - B\varepsilon_n(b, Bk).$$

令  $n \rightarrow \infty$  并取极限, 定理即得证.

定理 9 是定理 14 的特殊情形, 于定理 14 令  $A=1, B=1$ , 即得定理 9.

**例 1** 求和:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

**解** 于定理 14 令  $A=2, B=3, a=1, b=2, k=1$ , 则有

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)} = \sum_{n=1}^{\infty} \left( \frac{2}{2n-1} - \frac{3}{3n-1} \right) \\ = 2C(1, 2) - 3C(2, 3) + \ln \frac{2^2}{3}.$$

根据定理 6, 有

$$2C(1, 2) - 3C(2, 3) = \frac{\pi}{2\sqrt{3}} - \frac{1}{2} \ln 3.$$

于是  $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)} = \frac{\pi}{2\sqrt{3}} + 2 \ln 2 - \frac{3}{2} \ln 3.$



仿此求得

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-1)} = \sum_{n=1}^{\infty} \left( \frac{3}{3n-1} - \frac{4}{4n-1} \right)$$

$$= -\frac{\pi}{2\sqrt{3}} + \frac{\pi}{2} + \frac{3}{2} \ln 3 - 3 \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-1)} = \sum_{n=1}^{\infty} \left( \frac{2}{4n-1} - \frac{3}{6n-1} \right)$$

$$= \frac{\pi\sqrt{3}}{4} - \frac{\pi}{4} + \frac{1}{2} \ln 2 - \frac{3}{4} \ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-1)} = \sum_{n=1}^{\infty} \left( \frac{3}{6n-1} - \frac{4}{8n-1} \right)$$

$$= -\frac{\pi\sqrt{3}}{4} + \frac{\pi}{2\sqrt{2}} + \frac{\pi}{4} + \frac{3}{4} \ln 3 - \ln 2$$

$$- \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-1)(12n-1)} = \sum_{n=1}^{\infty} \left( \frac{2}{8n-1} - \frac{3}{12n-1} \right)$$

$$= \frac{\pi\sqrt{3}}{8} - \frac{\pi}{4\sqrt{2}} + \frac{\pi}{8} + \frac{1}{4} \ln 2 - \frac{3}{8} \ln 3$$

$$+ \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1) - \frac{\sqrt{3}}{4} \ln(2 + \sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)} - \frac{2}{(3n-1)(4n-1)} \right]$$

$$= \frac{\pi\sqrt{3}}{2} - \pi + 8 \ln 2 - \frac{9}{2} \ln 3,$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-1)(6n-1)} \\
& \sum_{n=1}^{\infty} \left[ \frac{1}{(3n-1)(4n-1)} - \frac{2}{(4n-1)(6n-1)} \right] \\
& = -\frac{2\pi}{\sqrt{3}} + \pi + 3 \ln 3 - 4 \ln 2, \\
& \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-1)(8n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-1)} - \frac{2}{(6n-1)(8n-1)} \right] \\
& = \frac{3\pi\sqrt{3}}{4} - \frac{\pi}{\sqrt{2}} - \frac{3\pi}{4} + \frac{5}{2} \ln 2 - \frac{9}{4} \ln 3 \\
& \quad + \sqrt{2} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-1)(12n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-1)(8n-1)} - \frac{2}{(8n-1)(12n-1)} \right] \\
& = -\frac{\pi\sqrt{3}}{2} + \frac{\pi}{\sqrt{2}} + \frac{3}{2} \ln 3 - \frac{3}{2} \ln 2 \\
& \quad + \frac{\sqrt{3}}{2} \ln(2 + \sqrt{3}) - \sqrt{2} \ln(\sqrt{2} + 1).
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)} \\
& = \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)(4n-1)} \right. \\
& \quad \left. - \frac{3}{(3n-1)(4n-1)(6n-1)} \right]
\end{aligned}$$

$$= \frac{5\pi\sqrt{3}}{4} - 2\pi + 10 \ln 2 - \frac{27}{4} \ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-1)(6n-1)(8n-1)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3}{(3n-1)(4n-1)(6n-1)} - \frac{8}{(4n-1)(6n-1)(8n-1)} \right]$$

$$= -\frac{8\pi\sqrt{3}}{5} + \frac{4\pi\sqrt{2}}{5} + \frac{9\pi}{5} - \frac{32}{5} \ln 2 + \frac{27}{5} \ln 3 - \frac{8\sqrt{2}}{5} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-1)(8n-1)(12n-1)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-1)(8n-1)} - \frac{3}{(6n-1)(8n-1)(12n-1)} \right]$$

$$= \frac{9\pi\sqrt{3}}{8} - \pi\sqrt{2} - \frac{3\pi}{8} + \frac{7}{2} \ln 2 - \frac{27}{8} \ln 3 + 2\sqrt{2} \ln(\sqrt{2}+1) - \frac{3\sqrt{3}}{4} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)}$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)} - \frac{4}{(3n-1)(4n-1)(6n-1)(8n-1)} \right]$$

$$\begin{aligned}
&= \frac{51\pi\sqrt{3}}{20} - \frac{16\pi\sqrt{2}}{15} - \frac{46\pi}{15} + \frac{178}{15} \ln 2 \\
&\quad - \frac{189}{20} \ln 3 + \frac{32\sqrt{3}}{15} \ln(\sqrt{2}+1), \\
&\sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(3n-1)(4n-1)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{4}{(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\
&= -\frac{61\pi}{10\sqrt{3}} + \frac{8\pi\sqrt{2}}{5} + \frac{11\pi}{10} - \frac{34}{5} \ln 2 + \frac{63}{10} \ln 3 \\
&\quad - \frac{16\sqrt{2}}{5} \ln(\sqrt{2}+1) + \sqrt{3} \ln(2+\sqrt{3}).
\end{aligned}$$

于是

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{6}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\
&= \frac{59\pi\sqrt{3}}{20} - \frac{32\pi\sqrt{2}}{15} - \frac{29\pi}{15} + \frac{158}{15} \ln 2 - \frac{189}{20} \ln 3 \\
&\quad + \frac{64\sqrt{2}}{15} \ln(\sqrt{2}+1) - \frac{6\sqrt{3}}{5} \ln(2+\sqrt{3}).
\end{aligned}$$

仿此求得

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(2n-1)(4n-1)(6n-1)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(4n-1)} - \frac{3}{(4n-1)(6n-1)} \right]
\end{aligned}$$

$$= -\frac{3\pi\sqrt{3}}{8} + \frac{\pi}{2} - \ln 2 + \frac{9}{8} \ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(6n-1)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)} - \frac{3}{(3n-1)(6n-1)} \right]$$

$$= -\frac{\pi}{4\sqrt{3}} + 2 \ln 2 - \frac{3}{4} \ln 3,$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(6n-1)(8n-1)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3}{(3n-1)(6n-1)} - \frac{8}{(6n-1)(8n-1)} \right]$$

$$= \frac{7\pi}{5\sqrt{3}} - \frac{2\pi\sqrt{2}}{5} - \frac{2\pi}{5} + \frac{6}{5} \ln 2 - \frac{6}{5} \ln 3$$

$$+ \frac{4\sqrt{2}}{5} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-1)(8n-1)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3}{(3n-1)(4n-1)} - \frac{8}{(4n-1)(8n-1)} \right]$$

$$= -\frac{\pi\sqrt{3}}{10} - \frac{\pi\sqrt{2}}{5} + \frac{3\pi}{10} - \frac{7}{5} \ln 2 + \frac{9}{10} \ln 3$$

$$+ \frac{2\sqrt{2}}{5} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(8n-1)(12n-1)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(8n-1)} - \frac{3}{(8n-1)(12n-1)} \right]$$

$$= -\frac{3\pi\sqrt{3}}{16} + \frac{\pi}{2\sqrt{2}} - \frac{3\pi}{16} - \frac{1}{2} \ln 2 + \frac{9}{16} \ln 3$$

$$\begin{aligned}
& -\frac{1}{\sqrt{2}} \ln(\sqrt{2}+1) + \frac{3\sqrt{3}}{8} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-1)(12n-1)} \\
& = \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-1)} - \frac{3}{(6n-1)(12n-1)} \right] \\
& = \frac{\pi\sqrt{3}}{8} - \frac{3\pi}{8} + \frac{1}{2} \ln 2 - \frac{3}{8} \ln 3 \\
& \quad + \frac{\sqrt{3}}{4} \ln(2+\sqrt{3}).
\end{aligned}$$

由此顺次求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(2n-1)(4n-1)(6n-1)(8n-1)} \\
& = \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(4n-1)(6n-1)} \right. \\
& \quad \left. - \frac{4}{(4n-1)(6n-1)(8n-1)} \right] \\
& = -\frac{9\pi\sqrt{3}}{8} + \frac{2\pi\sqrt{2}}{3} + \frac{7\pi}{6} - \frac{11}{3} \ln 2 \\
& \quad + \frac{27}{8} \ln 3 - \frac{4\sqrt{2}}{3} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(6n-1)(8n-1)} \\
& = \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)(6n-1)} \right. \\
& \quad \left. - \frac{4}{(3n-1)(6n-1)(8n-1)} \right] \\
& = -\frac{13\pi\sqrt{3}}{20} + \frac{8\pi\sqrt{2}}{15} + \frac{8\pi}{15} - \frac{14}{15} \ln 2 + \frac{27}{20} \ln 3
\end{aligned}$$

$$-\frac{16\sqrt{2}}{15} \ln(\sqrt{2}+1),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(3n-1)(6n-1)(8n-1)(12n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(3n-1)(6n-1)(8n-1)} \right. \\ & \quad \left. - \frac{4}{(6n-1)(8n-1)(12n-1)} \right] \\ &= \frac{37\pi}{15\sqrt{3}} - \frac{4\pi\sqrt{2}}{5} - \frac{2\pi}{15} + \frac{12}{5} \ln 2 - \frac{12}{5} \ln 3 \\ & \quad + \frac{8\sqrt{2}}{5} \ln(\sqrt{2}+1) - \frac{2}{\sqrt{3}} \ln(2+\sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(4n-1)(8n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)(4n-1)} \right. \\ & \quad \left. - \frac{4}{(3n-1)(4n-1)(8n-1)} \right] \\ &= \frac{3\pi\sqrt{3}}{10} + \frac{4\pi\sqrt{2}}{15} - \frac{11\pi}{15} + \frac{68}{15} \ln 2 - \frac{27}{10} \ln 3 \\ & \quad - \frac{8\sqrt{2}}{15} \ln(\sqrt{2}+1), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-1)(8n-1)(12n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(3n-1)(4n-1)(8n-1)} \right. \\ & \quad \left. - \frac{4}{(4n-1)(8n-1)(12n-1)} \right] \\ &= \frac{13\pi}{20\sqrt{3}} - \frac{2\pi\sqrt{2}}{5} + \frac{7\pi}{20} + \frac{1}{5} \ln 2 - \frac{9}{20} \ln 3 \end{aligned}$$

$$\begin{aligned}
& + \frac{4\sqrt{2}}{5} \ln(\sqrt{2} + 1) - \frac{\sqrt{3}}{2} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-1)(6n-1)(12n-1)} \\
& = \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(3n-1)(4n-1)(6n-1)} \right. \\
& \quad \left. - \frac{4}{(4n-1)(6n-1)(12n-1)} \right] \\
& = -\frac{7\pi}{6\sqrt{3}} + \frac{5\pi}{6} + \frac{3}{2} \ln 3 - 2 \ln 2 \\
& \quad - \frac{1}{\sqrt{3}} \ln(2 + \sqrt{3}); \\
& \sum_{n=1}^{\infty} \frac{1}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \\
& = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(4n-1)(6n-1)(8n-1)} \right. \\
& \quad \left. - \frac{6}{(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\
& = -\frac{63\pi\sqrt{3}}{40} + \frac{4\pi\sqrt{2}}{3} + \frac{41\pi}{60} - \frac{74}{15} \ln 2 + \frac{189}{40} \ln 3 \\
& \quad - \frac{8\sqrt{2}}{3} \ln(\sqrt{2} + 1) + \frac{9\sqrt{3}}{10} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)} \\
& = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)(6n-1)(8n-1)} \right. \\
& \quad \left. - \frac{6}{(3n-1)(6n-1)(8n-1)(12n-1)} \right]
\end{aligned}$$



$$= -\frac{67\pi}{20\sqrt{3}} + \frac{16\pi\sqrt{2}}{15} + \frac{4\pi}{15} - \frac{46}{15} \ln 2 + \frac{63}{20} \ln 3 \\ - \frac{32\sqrt{2}}{15} \ln(\sqrt{2} + 1) + \frac{4\sqrt{3}}{5} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)(4n-1)(8n-1)} \right.$$

$$\left. - \frac{6}{(3n-1)(4n-1)(8n-1)(12n-1)} \right]$$

$$= -\frac{\pi\sqrt{3}}{5} + \frac{8\pi\sqrt{2}}{15} - \frac{17\pi}{30} + \frac{2}{3} \ln 2$$

$$- \frac{16\sqrt{2}}{15} \ln(\sqrt{2} + 1) + \frac{3\sqrt{3}}{5} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)} \right.$$

$$\left. - \frac{6}{(3n-1)(4n-1)(6n-1)(12n-1)} \right]$$

$$= \frac{43\pi}{20\sqrt{3}} - \frac{7\pi}{5} + \frac{22}{5} \ln 2 - \frac{63}{20} \ln 3$$

$$+ \frac{2\sqrt{3}}{5} \ln(2 + \sqrt{3}).$$

例2 求和:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-1)(4n-1)},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-1)(4n-1)(6n-1)},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)}.$$

解 于定理 14 令  $A=4$ ,  $B=6$ ,  $a=1$ ,  $b=2$ ,  $k=1$ , 则有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-4)} &= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{4}{4n-3} - \frac{3}{3n-2} \right) \\ &= 2C(1,4) - \frac{3}{2} C(1,3) + \frac{1}{2} \ln \frac{2^2}{3}. \end{aligned}$$

根据定理 6, 有

$$2C(1,4) - \frac{3}{2} C(1,3) = -\frac{\pi}{4\sqrt{3}} + \frac{\pi}{4} + \frac{1}{4} \ln \frac{2^2}{3}.$$

$$\begin{aligned} \text{于是 } \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-4)} &= -\frac{\pi}{4\sqrt{3}} + \frac{\pi}{4} + \frac{3}{2} \ln 2 \\ &\quad - \frac{3}{4} \ln 3. \end{aligned}$$

仿此推得

$$\sum_{n=1}^{\infty} \frac{1}{(6n-4)(8n-5)} = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{3}{3n-2} - \frac{8}{8n-5} \right)$$

$$\begin{aligned} &= \frac{\pi}{4\sqrt{3}} - \frac{\pi}{2\sqrt{2}} + \frac{\pi}{4} + \frac{3}{4} \ln 3 - 2 \ln 2 \\ &\quad + \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-5)(12n-7)} = \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{8}{8n-5} - \frac{12}{12n-7} \right)$$

$$\begin{aligned} &= \frac{\pi\sqrt{3}}{8} + \frac{\pi}{4\sqrt{2}} - \frac{3\pi}{8} + \frac{1}{4} \ln 2 - \frac{3}{8} \ln 3 \\ &\quad - \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1) + \frac{\sqrt{3}}{4} \ln(2+\sqrt{3}). \end{aligned}$$

由此顺次求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-4)(8n-5)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-4)} - \frac{2}{(6n-4)(8n-5)} \right] \\
&= -\frac{\pi\sqrt{3}}{4} + \frac{\pi}{\sqrt{2}} - \frac{\pi}{4} + \frac{11}{2} \ln 2 - \frac{9}{4} \ln 3 \\
&\quad - \sqrt{2} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-4)(8n-5)(12n-7)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-5)} - \frac{2}{(8n-5)(12n-7)} \right] \\
&= -\frac{\pi}{2\sqrt{3}} - \frac{\pi}{\sqrt{2}} + \pi + \frac{3}{2} \ln 3 - \frac{5}{2} \ln 2 \\
&\quad + \sqrt{2} \ln(\sqrt{2}+1) - \frac{\sqrt{3}}{2} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-4)(8n-5)(12n-7)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-4)(8n-5)} \right. \\
&\quad \left. - \frac{3}{(6n-4)(8n-5)(12n-7)} \right] \\
&= \frac{\pi\sqrt{3}}{8} + \pi\sqrt{2} - \frac{13\pi}{8} + \frac{13}{2} \ln 2 - \frac{27}{8} \ln 3 \\
&\quad - 2\sqrt{2} \ln(\sqrt{2}+1) + \frac{3\sqrt{3}}{4} \ln(2+\sqrt{3}).
\end{aligned}$$

于是

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-1)(4n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-4)(8n-5)} \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{(4n-1)(6n-1)(8n-1)} \Big] \\
& = -\pi\sqrt{3} + \pi\sqrt{2} + \frac{\pi}{2} + 3\ln 2 - 2\sqrt{2}\ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-1)(4n-1)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-5)(12n-7)} \right. \\
& \quad \left. - \frac{1}{(6n-1)(8n-1)(12n-1)} \right] \\
& = \frac{\pi}{\sqrt{3}} - \pi\sqrt{2} + \pi - \ln 2 + 2\sqrt{2}\ln(\sqrt{2}+1) \\
& \quad - \sqrt{3}\ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-1)(4n-1)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-4)(8n-5)(12n-7)} \right. \\
& \quad \left. - \frac{1}{(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\
& = -\pi\sqrt{3} + 2\pi\sqrt{2} - \frac{5\pi}{4} + 3\ln 2 \\
& \quad - 4\sqrt{2}\ln(\sqrt{2}+1) + \frac{3\sqrt{3}}{2}\ln(2+\sqrt{3}).
\end{aligned}$$

**例3** 求和:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(4n-1)(6n-1)}.$$

**解** 于定理14令  $A=6$ ,  $B=8$ ,  $a=1$ ,  $b=3$ ,  $k=1$ ,  
即可推得

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(6n-5)(8n-5)} &= \frac{1}{10} \sum_{n=1}^{\infty} \left( \frac{6}{6n-5} - \frac{8}{8n-5} \right) \\ &= \frac{\pi\sqrt{3}}{20} - \frac{\pi}{10\sqrt{2}} + \frac{\pi}{20} + \frac{3}{20} \ln 3 - \frac{1}{5} \ln 2 \\ &\quad + \frac{1}{5\sqrt{2}} \ln(\sqrt{2}+1).\end{aligned}$$

仿此推得

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(8n-5)(12n-7)} &= \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{8}{8n-5} - \frac{12}{12n-7} \right) \\ &= \frac{\pi\sqrt{3}}{8} + \frac{\pi}{4\sqrt{2}} - \frac{3\pi}{8} + \frac{1}{4} \ln 2 - \frac{3}{8} \ln 3 \\ &\quad - \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1) + \frac{\sqrt{3}}{4} \ln(2+\sqrt{3}).\end{aligned}$$

于是

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(6n-5)(8n-5)(12n-7)} &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-5)} - \frac{2}{(8n-5)(12n-7)} \right] \\ &= -\frac{\pi}{5\sqrt{3}} - \frac{\pi}{5\sqrt{2}} + \frac{4\pi}{15} + \frac{3}{10} \ln 3 - \frac{7}{30} \ln 2 \\ &\quad + \frac{\sqrt{2}}{5} \ln(\sqrt{2}+1) - \frac{1}{2\sqrt{3}} \ln(2+\sqrt{3}).\end{aligned}$$

于定理14令  $A=6$ ,  $B=8$ ,  $a=4$ ,  $b=7$ ,  $k=1$ , 即可推得

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(6n-2)(8n-1)} &= \frac{1}{10} \sum_{n=1}^{\infty} \left( \frac{3}{3n-1} - \frac{8}{8n-1} \right) \\ &= -\frac{\pi}{20\sqrt{3}} + \frac{\pi}{10\sqrt{2}} + \frac{\pi}{20} + \frac{3}{20} \ln 3 - \frac{2}{5} \ln 2 \\ &\quad - \frac{1}{5\sqrt{2}} \ln(\sqrt{2}+1).\end{aligned}$$

仿此推得

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(8n-1)(12n-1)} &= \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{8}{8n-1} - \frac{12}{12n-1} \right) \\ &= \frac{\pi\sqrt{3}}{8} - \frac{\pi}{4\sqrt{2}} + \frac{\pi}{8} + \frac{1}{4} \ln 2 - \frac{3}{8} \ln 3 \\ &\quad + \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1) - \frac{\sqrt{3}}{4} \ln(2+\sqrt{3}).\end{aligned}$$

于是

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(6n-2)(8n-1)(12n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-2)(8n-1)} - \frac{2}{(8n-1)(12n-1)} \right] \\ &= -\frac{4\pi}{15\sqrt{3}} + \frac{\pi}{5\sqrt{2}} - \frac{\pi}{15} + \frac{3}{10} \ln 3 - \frac{3}{10} \ln 2 \\ &\quad - \frac{\sqrt{2}}{5} \ln(\sqrt{2}+1) + \frac{1}{2\sqrt{3}} \ln(2+\sqrt{3}).\end{aligned}$$

从而

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(4n-1)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-5)(12n-7)} \right. \\ &\quad \left. - \frac{1}{(6n-2)(8n-1)(12n-1)} \right] \\ &= \frac{\pi}{15\sqrt{3}} - \frac{\pi\sqrt{2}}{5} + \frac{\pi}{3} + \frac{1}{15} \ln 2 + \frac{2\sqrt{2}}{5} \ln(\sqrt{2}+1) \\ &\quad - \frac{1}{\sqrt{3}} \ln(2+\sqrt{3}).\end{aligned}$$

从以上的结果还可推得

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(3n-2)(4n-1)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-5)(12n-7)} \right.\end{aligned}$$

$$+ \frac{1}{(6n-2)(8n-1)(12n-1)} \Big] \\ = -\frac{7\pi}{15\sqrt{3}} + \frac{\pi}{5} + \frac{3}{5} \ln 3 - \frac{8}{15} \ln 2.$$

例4 求和:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(4n-3)(4n-1)}.$$

解 于定理 14 令  $A=4$ ,  $B=8$ ,  $a=1$ ,  $b=1$ ,  $k=1$ , 即可推得

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(4n-3)(8n-7)} &= -\frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{4}{4n-3} - \frac{8}{8n-7} \right) \\ &= \frac{\pi}{4\sqrt{2}} + \frac{1}{4} \ln 2 + \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1). \end{aligned}$$

仿此推得

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)} &= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{8n-7} - \frac{1}{8n-5} \right) \\ &= \frac{\pi}{16} + \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(4n-1)(8n-3)} &= -\frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{4}{4n-1} - \frac{8}{8n-3} \right) \\ &= -\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4} + \frac{1}{4} \ln 2 - \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(8n-3)(8n-1)} &= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{8n-3} - \frac{1}{8n-1} \right) \\ &= \frac{\pi}{16} - \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1). \end{aligned}$$

于是 
$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(8n-7)(8n-5)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(8n-7)} - \frac{2}{(8n-7)(8n-5)} \right]$$

$$= \frac{\pi}{4\sqrt{2}} - \frac{\pi}{8} + \frac{1}{4} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(8n-3)(8n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(8n-3)} - \frac{2}{(8n-3)(8n-1)} \right]$$

$$= -\frac{\pi}{4\sqrt{2}} + \frac{\pi}{8} + \frac{1}{4} \ln 2.$$

从而  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(4n-3)(4n-1)}$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(8n-7)(8n-5)} - \frac{1}{(4n-1)(8n-3)(8n-1)} \right]$$

$$= \frac{\pi}{2\sqrt{2}} - \frac{\pi}{4}.$$

例5 求和:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(3n-1)},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)(4n-1)},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)(4n-1)(6n-1)},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)(6n-1)},$$



解 于定理 14 令  $A=4$ ,  $B=6$ ,  $a=1$ ,  $b=1$ ,  $k=1$ ,  
即可推得

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)} &= -\frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{4}{4n-3} - \frac{6}{6n-5} \right) \\ &= \frac{\pi\sqrt{3}}{4} - \frac{\pi}{4} + \frac{3}{4} \ln 3 - \frac{1}{2} \ln 2.\end{aligned}$$

仿此推得

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)} &= \sum_{n=1}^{\infty} \left( \frac{1}{6n-5} - \frac{1}{6n-4} \right) \\ &= \frac{\pi}{6\sqrt{3}} + \frac{1}{3} \ln 2,\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(6n-4)(8n-5)} &= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{3}{3n-2} - \frac{8}{8n-5} \right) \\ &= \frac{\pi}{4\sqrt{3}} - \frac{\pi}{2\sqrt{2}} + \frac{\pi}{4} + \frac{3}{4} \ln 3 - 2 \ln 2 \\ &\quad + \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1),\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(8n-5)(12n-7)} &= \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{8}{8n-5} - \frac{12}{12n-7} \right) \\ &= \frac{\pi\sqrt{3}}{8} + \frac{\pi}{4\sqrt{2}} - \frac{3\pi}{8} + \frac{1}{4} \ln 2 - \frac{3}{8} \ln 3 \\ &\quad - \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1) + \frac{\sqrt{3}}{4} \ln(2+\sqrt{3}),\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-2)} &= -\frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{4}{4n-1} - \frac{6}{6n-2} \right) \\ &= -\frac{\pi}{4\sqrt{3}} + \frac{\pi}{4} + \frac{3}{4} \ln 3 - \frac{3}{2} \ln 2,\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)} = \sum_{n=1}^{\infty} \left( \frac{1}{6n-2} - \frac{1}{6n-1} \right)$$

$$= \frac{\pi}{6\sqrt{3}} - \frac{1}{3} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-1)} = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{6}{6n-1} - \frac{8}{8n-1} \right)$$

$$= -\frac{\pi\sqrt{3}}{4} + \frac{\pi}{2\sqrt{2}} + \frac{\pi}{4} + \frac{3}{4} \ln 3 - \ln 2$$

$$- \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-1)(12n-1)} = \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{8}{8n-1} - \frac{12}{12n-1} \right)$$

$$= \frac{\pi\sqrt{3}}{8} - \frac{\pi}{4\sqrt{2}} + \frac{\pi}{8} + \frac{1}{4} \ln 2 - \frac{3}{8} \ln 3$$

$$+ \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1) - \frac{\sqrt{3}}{4} \ln(2 + \sqrt{3}).$$

由此顺次求得

$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(6n-4)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2}{(4n-3)(6n-5)} - \frac{3}{(6n-5)(6n-4)} \right]$$

$$= \frac{\pi}{\sqrt{3}} - \frac{\pi}{2} + \frac{3}{2} \ln 3 - 2 \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-5)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3}{(6n-5)(6n-4)} - \frac{4}{(6n-4)(8n-5)} \right]$$

$$= -\frac{\pi}{10\sqrt{3}} + \frac{\pi\sqrt{2}}{5} - \frac{\pi}{5} + \frac{9}{5} \ln 2 - \frac{3}{5} \ln 3$$

$$- \frac{2\sqrt{2}}{5} \ln(\sqrt{2} + 1),$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-4)(8n-5)(12n-7)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-5)} - \frac{2}{(8n-5)(12n-7)} \right] \\
&= -\frac{\pi}{2\sqrt{3}} - \frac{\pi}{\sqrt{2}} + \pi + \frac{3}{2} \ln 3 - \frac{5}{2} \ln 2 \\
&\quad + \sqrt{2} \ln(\sqrt{2} + 1) - \frac{\sqrt{3}}{2} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-2)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2}{(4n-1)(6n-2)} - \frac{3}{(6n-2)(6n-1)} \right] \\
&= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2} + \frac{3}{2} \ln 3 - 2 \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)(8n-1)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3}{(6n-2)(6n-1)} - \frac{4}{(6n-1)(8n-1)} \right] \\
&= \frac{7\pi}{10\sqrt{3}} - \frac{\pi\sqrt{2}}{5} - \frac{\pi}{5} + \frac{3}{5} \ln 2 - \frac{3}{5} \ln 3 \\
&\quad + \frac{2\sqrt{2}}{5} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-1)(12n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-1)(8n-1)} - \frac{2}{(8n-1)(12n-1)} \right] \\
&= -\frac{\pi\sqrt{3}}{2} + \frac{\pi}{\sqrt{2}} + \frac{3}{2} \ln 3 - \frac{3}{2} \ln 2 \\
&\quad - \sqrt{2} \ln(\sqrt{2} + 1) + \frac{\sqrt{3}}{2} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)} \right. \\
&\quad \left. - \frac{2}{(6n-5)(6n-4)(8n-5)} \right] \\
&= \frac{2\pi\sqrt{3}}{5} - \frac{2\pi\sqrt{2}}{5} - \frac{\pi}{10} + \frac{27}{10} \ln 3 - \frac{28}{5} \ln 2 \\
&\quad + \frac{4\sqrt{2}}{5} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-5)(12n-7)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-5)} \right. \\
&\quad \left. - \frac{2}{(6n-4)(8n-5)(12n-7)} \right] \\
&= \frac{\pi\sqrt{3}}{10} + \frac{2\pi\sqrt{2}}{5} - \frac{11\pi}{15} + \frac{34}{15} \ln 2 - \frac{6}{5} \ln 3 \\
&\quad - \frac{4\sqrt{2}}{5} \ln(\sqrt{2}+1) + \frac{1}{\sqrt{3}} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-2)(6n-1)} \right. \\
&\quad \left. - \frac{2}{(6n-2)(6n-1)(8n-1)} \right] \\
&= -\frac{4\pi\sqrt{3}}{5} + \frac{2\pi\sqrt{2}}{5} + \frac{9\pi}{10} + \frac{27}{10} \ln 3 - \frac{16}{5} \ln 2 \\
&\quad - \frac{4\sqrt{2}}{5} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)(8n-1)(12n-1)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-2)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{2}{(6n-1)(8n-1)(12n-1)} \right] \\
&= \frac{37\pi}{30\sqrt{3}} - \frac{2\pi\sqrt{2}}{5} - \frac{\pi}{15} + \frac{6}{5} \ln 2 - \frac{6}{5} \ln 3 \\
&\quad + \frac{4\sqrt{2}}{5} \ln(\sqrt{2}+1) - \frac{1}{\sqrt{3}} \ln(2+\sqrt{3}).
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)(12n-7)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)} \right. \\
&\quad \left. - \frac{3}{(6n-5)(6n-4)(8n-5)(12n-7)} \right] \\
&= \frac{\pi\sqrt{3}}{20} - \frac{4\pi\sqrt{2}}{5} + \frac{21\pi}{20} + \frac{63}{20} \ln 3 - \frac{31}{5} \ln 2 \\
&\quad + \frac{8\sqrt{2}}{5} \ln(\sqrt{2}+1) - \frac{\sqrt{3}}{2} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)(12n-1)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{3}{(6n-2)(6n-1)(8n-1)(12n-1)} \right] \\
&= -\frac{61\pi}{20\sqrt{3}} + \frac{4\pi\sqrt{2}}{5} + \frac{11\pi}{20} + \frac{63}{20} \ln 3 - \frac{17}{5} \ln 2
\end{aligned}$$

$$-\frac{8\sqrt{2}}{5} \ln(\sqrt{2}+1) + \frac{\sqrt{3}}{2} \ln(2+\sqrt{3}).$$

于是

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(3n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)} - \frac{1}{(4n-1)(6n-2)(6n-1)} \right]$$

$$= \frac{2\pi}{\sqrt{3}} - \pi,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-5)} - \frac{1}{(6n-2)(6n-1)(8n-1)} \right]$$

$$= -\frac{4\pi}{5\sqrt{3}} + \frac{2\pi\sqrt{2}}{5} + \frac{6}{5} \ln 2$$

$$- \frac{4\sqrt{2}}{5} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)} - \frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)} \right]$$

$$= \frac{6\pi\sqrt{3}}{5} - \frac{4\pi\sqrt{2}}{5} - \pi - \frac{12}{5} \ln 2$$

$$\begin{aligned}
& + \frac{8\sqrt{2}}{5} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)(4n-1)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-5)(12n-7)} \right. \\
& \quad \left. - \frac{1}{(6n-2)(6n-1)(8n-1)(12n-1)} \right] \\
& = -\frac{14\pi}{15\sqrt{3}} + \frac{4\pi\sqrt{2}}{5} - \frac{2\pi}{3} + \frac{16}{15} \ln 2 \\
& \quad - \frac{8\sqrt{2}}{5} \ln(\sqrt{2} + 1) + \frac{2}{\sqrt{3}} \ln(2 + \sqrt{3}), \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)(12n-7)} \right. \\
& \quad \left. - \frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)(12n-1)} \right] \\
& = \frac{16\pi}{5\sqrt{3}} - \frac{8\pi\sqrt{2}}{5} + \frac{\pi}{2} - \frac{14}{5} \ln 2 \\
& \quad + \frac{16\sqrt{2}}{5} \ln(\sqrt{2} + 1) - \sqrt{3} \ln(2 + \sqrt{3}).
\end{aligned}$$

从以上的结果还可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(3n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)} \right. \\
& \quad \left. + \frac{1}{(4n-1)(6n-2)(6n-1)} \right]
\end{aligned}$$

$$= 3 \ln 3 - 4 \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-5)} + \frac{1}{(6n-2)(6n-1)(8n-1)} \right]$$

$$= \frac{\pi\sqrt{3}}{5} - \frac{2\pi}{5} + \frac{12}{5} \ln 2 - \frac{6}{5} \ln 3;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)} + \frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)} \right]$$

$$= -\frac{2\pi\sqrt{3}}{5} + \frac{4\pi}{5} + \frac{27}{5} \ln 3 - \frac{44}{5} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(4n-1)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-5)(12n-7)} + \frac{1}{(6n-2)(6n-1)(8n-1)(12n-1)} \right]$$

$$= \frac{23\pi}{15\sqrt{3}} - \frac{4\pi}{5} + \frac{52}{15} \ln 2 - \frac{12}{5} \ln 3;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(3n-1)(4n-1)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)(12n-7)} \right]$$



$$+ \frac{1}{(4n-1)(6n-2)(6n-1)(8n-1)(12n-1)} \Bigg] \\ = -\frac{29\pi}{10\sqrt{3}} + \frac{8\pi}{5} + \frac{63}{10} \ln 3 - \frac{48}{5} \ln 2.$$

仿此求得

$$(一) \quad \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(8n-7)} \\ = -\frac{3\pi\sqrt{3}}{4} + \frac{\pi}{\sqrt{2}} + \frac{3\pi}{4} + \frac{5}{2} \ln 2 - \frac{9}{4} \ln 3 \\ + \sqrt{2} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(8n-7)(8n-5)} \\ = -\frac{3\pi\sqrt{3}}{20} + \frac{3\pi}{10\sqrt{2}} + \frac{\pi}{10} + \frac{3}{5} \ln 2 - \frac{9}{20} \ln 3 \\ + \frac{\sqrt{2}}{5} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(12n-7)} \\ = -\frac{3\pi\sqrt{3}}{56} - \frac{3\pi}{28\sqrt{2}} + \frac{5\pi}{28} - \frac{3}{28} \ln 2 \\ + \frac{9}{56} \ln 3 + \frac{\sqrt{2}}{7} \ln(\sqrt{2}+1) \\ - \frac{3\sqrt{3}}{28} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-2)(8n-3)} \\ = \frac{\pi\sqrt{3}}{4} - \frac{\pi}{\sqrt{2}} + \frac{\pi}{4} + \frac{11}{2} \ln 2 - \frac{9}{4} \ln 3 \\ - \sqrt{2} \ln(\sqrt{2}+1),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(6n-2)(8n-3)(8n-1)} \\ &= \frac{\pi\sqrt{3}}{20} - \frac{3\pi}{10\sqrt{2}} + \frac{\pi}{10} + \frac{6}{5} \ln 2 - \frac{9}{20} \ln 3 \\ & \quad - \frac{\sqrt{2}}{5} \ln(\sqrt{2}+1), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(8n-3)(8n-1)(12n-1)} \\ &= -\frac{3\pi\sqrt{3}}{56} + \frac{3\pi}{28\sqrt{2}} - \frac{\pi}{28} - \frac{3}{28} \ln 2 + \frac{9}{56} \ln 3 \\ & \quad - \frac{\sqrt{2}}{7} \ln(\sqrt{2}+1) + \frac{3\sqrt{3}}{28} \ln(2+\sqrt{3}). \end{aligned}$$

由此顺次求得

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(4n-3)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)} \right. \\ & \quad \left. - \frac{1}{(4n-1)(6n-2)(8n-3)} \right] \\ &= -\pi\sqrt{3} + \pi\sqrt{2} + \frac{\pi}{2} - 3 \ln 2 + 2\sqrt{2} \ln(\sqrt{2}+1), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(4n-3)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)} \right. \\ & \quad \left. + \frac{1}{(4n-1)(6n-2)(8n-3)} \right] \\ &= -\frac{\pi\sqrt{3}}{2} + \pi + 8 \ln 2 - \frac{9}{2} \ln 3, \end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(4n-3)(4n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-7)(8n-5)} \right. \\
&\quad \left. - \frac{1}{(6n-2)(8n-3)(8n-1)} \right] \\
&= -\frac{\pi\sqrt{3}}{5} + \frac{3\pi}{5\sqrt{2}} - \frac{3}{5} \ln 2 + \frac{2\sqrt{2}}{5} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(3n-2)(4n-3)(4n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-7)(8n-5)} \right. \\
&\quad \left. + \frac{1}{(6n-2)(8n-3)(8n-1)} \right] \\
&= -\frac{\pi\sqrt{3}}{10} + \frac{\pi}{5} + \frac{9}{5} \ln 2 - \frac{9}{10} \ln 3,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(4n-3)(4n-1)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-5)(12n-7)} \right. \\
&\quad \left. - \frac{1}{(8n-3)(8n-1)(12n-1)} \right] \\
&= -\frac{3\pi}{14\sqrt{2}} + \frac{3\pi}{14} + \frac{2\sqrt{2}}{7} \ln(\sqrt{2}+1) \\
&\quad - \frac{3\sqrt{3}}{14} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-1)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-5)(12n-7)} \right.
\end{aligned}$$

$$+ \frac{1}{(8n-3)(8n-1)(12n-1)} \Big] \\ = -\frac{3\pi\sqrt{3}}{28} + \frac{\pi}{7} - \frac{3}{14} \ln 2 + \frac{9}{28} \ln 3.$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)} \\ = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)} \right. \\ \left. - \frac{2}{(6n-5)(8n-7)(8n-5)} \right] \\ = -\frac{9\pi\sqrt{3}}{20} + \frac{\pi\sqrt{2}}{5} + \frac{11\pi}{20} + \frac{13}{10} \ln 2 \\ - \frac{27}{20} \ln 3 + \frac{3\sqrt{2}}{5} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(8n-7)(8n-5)(12n-7)} \\ = \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-7)(8n-5)} \right. \\ \left. - \frac{2}{(8n-7)(8n-5)(12n-7)} \right] \\ = -\frac{\pi\sqrt{3}}{70} + \frac{3\pi\sqrt{2}}{35} - \frac{3\pi}{35} + \frac{18}{70} \ln 2 - \frac{9}{35} \ln 3 \\ - \frac{\sqrt{2}}{35} \ln(\sqrt{2}+1) + \frac{\sqrt{3}}{14} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)} \\ = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-2)(8n-3)} \right.$$

$$\begin{aligned}
& - \frac{2}{(6n-2)(8n-3)(8n-1)} \Big] \\
& = \frac{3\pi\sqrt{3}}{20} - \frac{\pi\sqrt{2}}{5} + \frac{\pi}{20} + \frac{31}{10} \ln 2 - \frac{27}{20} \ln 3 \\
& \quad - \frac{3\sqrt{2}}{5} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-2)(8n-3)(8n-1)(12n-1)} \\
& = \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-2)(8n-3)(8n-1)} \right. \\
& \quad \left. - \frac{2}{(8n-3)(8n-1)(12n-1)} \right] \\
& = \frac{11\pi}{70\sqrt{3}} - \frac{3\pi\sqrt{2}}{35} + \frac{2\pi}{35} + \frac{33}{70} \ln 2 - \frac{9}{35} \ln 3 \\
& \quad + \frac{\sqrt{2}}{35} \ln(\sqrt{2}+1) - \frac{\sqrt{2}}{14} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(4n-3)(4n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)} \right. \\
& \quad \left. - \frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)} \right] \\
& = -\frac{3\pi\sqrt{3}}{5} + \frac{2\pi\sqrt{2}}{5} + \frac{\pi}{2} - \frac{9}{5} \ln 2 \\
& \quad + \frac{6\sqrt{2}}{5} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(4n-3)(4n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)} \right.
\end{aligned}$$

$$+ \frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)} \Bigg] \\ = -\frac{3\pi\sqrt{3}}{10} + \frac{3\pi}{5} + \frac{22}{5} \ln 2 - \frac{27}{10} \ln 3,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(4n-3)(4n-1)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-7)(8n-5)(12n-7)} \right. \\ \left. - \frac{1}{(6n-2)(8n-3)(8n-1)(12n-1)} \right]$$

$$= -\frac{\pi}{5\sqrt{3}} + \frac{6\pi\sqrt{2}}{35} - \frac{\pi}{7} - \frac{1}{5} \ln 2$$

$$+ \frac{2\sqrt{2}}{35} \ln(\sqrt{2}+1) + \frac{\sqrt{3}}{7} \ln(2+\sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(4n-3)(4n-1)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-7)(8n-5)(12n-7)} \right. \\ \left. + \frac{1}{(6n-2)(8n-3)(8n-1)(12n-1)} \right]$$

$$= \frac{4\pi}{35\sqrt{3}} - \frac{\pi}{35} + \frac{26}{35} \ln 2 - \frac{18}{35} \ln 3.$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)(12n-7)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)} \right. \\ \left. - \frac{3}{(6n-5)(8n-7)(8n-5)(12n-7)} \right]$$

$$= -\frac{57\pi\sqrt{3}}{280} - \frac{\pi\sqrt{2}}{35} + \frac{113\pi}{280} + \frac{17}{70} \ln 2 - \frac{81}{280} \ln 3 \\ + \frac{12\sqrt{2}}{35} \ln(\sqrt{2} + 1) - \frac{3\sqrt{3}}{28} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-7)(8n-3)(8n-1)(12n-1)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)} \right. \\ \left. - \frac{3}{(6n-2)(8n-3)(8n-1)(12n-1)} \right]$$

$$= -\frac{\pi\sqrt{3}}{280} + \frac{\pi\sqrt{2}}{35} - \frac{17\pi}{280} + \frac{59}{70} \ln 2 - \frac{81}{280} \ln 3 \\ - \frac{12\sqrt{2}}{35} \ln(\sqrt{2} + 1) + \frac{3\sqrt{3}}{28} \ln(2 + \sqrt{3});$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(4n-3)(4n-1)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)(12n-7)} \right. \\ \left. - \frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)(12n-1)} \right]$$

$$= -\frac{\pi\sqrt{3}}{5} - \frac{2\pi\sqrt{2}}{35} + \frac{13\pi}{28} - \frac{3}{5} \ln 2 \\ + \frac{24\sqrt{2}}{35} \ln(\sqrt{2} + 1) - \frac{3\sqrt{3}}{14} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(4n-3)(4n-1)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(8n-5)(12n-7)} \right. \\ \left. + \frac{1}{(4n-1)(6n-2)(8n-3)(8n-1)(12n-1)} \right]$$

$$= -\frac{29\pi\sqrt{3}}{140} + \frac{12\pi}{35} + \frac{38}{35} \ln 2 - \frac{81}{140} \ln 3.$$

$$\begin{aligned}
 (二) \quad & \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)} \\
 &= \frac{9\pi\sqrt{3}}{8} - \pi\sqrt{2} - \frac{3\pi}{8} + \frac{27}{8} \ln 3 - \frac{7}{2} \ln 2 \\
 &\quad - 2\sqrt{2} \ln(\sqrt{2}+1) + \frac{3\sqrt{3}}{4} \ln(2+\sqrt{3}), \\
 & \sum_{n=1}^{\infty} \frac{1}{(6n-5)(8n-7)(12n-11)(12n-7)} \\
 &= \frac{25\pi}{56\sqrt{3}} - \frac{\pi\sqrt{2}}{7} - \frac{\pi}{42} + \frac{3}{7} \ln 3 - \frac{19}{42} \ln 2 \\
 &\quad + \frac{11}{28\sqrt{3}} \ln(2+\sqrt{3}) - \frac{2\sqrt{2}}{7} \ln(\sqrt{2}+1), \\
 & \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)} \\
 &= \frac{\pi\sqrt{3}}{8} + \pi\sqrt{2} - \frac{13\pi}{8} + \frac{27}{8} \ln 3 - \frac{13}{2} \ln 2 \\
 &\quad + 2\sqrt{2} \ln(\sqrt{2}+1) - \frac{3\sqrt{3}}{4} \ln(2+\sqrt{3}), \\
 & \sum_{n=1}^{\infty} \frac{1}{(6n-2)(8n-3)(12n-5)(12n-1)} \\
 &= \frac{19\pi}{168\sqrt{3}} + \frac{\pi\sqrt{2}}{7} - \frac{11\pi}{42} + \frac{3}{7} \ln 3 - \frac{11}{14} \ln 2 \\
 &\quad + \frac{2\sqrt{2}}{7} \ln(\sqrt{2}+1) - \frac{11}{28\sqrt{3}} \ln(2+\sqrt{3}).
 \end{aligned}$$

由此求得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(4n-3)(6n-5)}$$



$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)} \right. \\
&\quad \left. - \frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)} \right] \\
&= \pi \sqrt{3} - 2\pi \sqrt{2} + \frac{5\pi}{4} + 3 \ln 2 - 4\sqrt{2} \ln(\sqrt{2} + 1) \\
&\quad + \frac{3\sqrt{3}}{2} \ln(2 + \sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(4n-3)(6n-5)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)} \right. \\
&\quad \left. + \frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)} \right] \\
&= \frac{5\pi\sqrt{3}}{4} - 2\pi + \frac{27}{4} \ln 3 - 10 \ln 2,
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(4n-3)(6n-5)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-7)(12n-11)(12n-7)} \right. \\
&\quad \left. - \frac{1}{(6n-2)(8n-3)(12n-5)(12n-1)} \right] \\
&= \frac{\pi}{3\sqrt{3}} - \frac{2\pi\sqrt{2}}{7} + \frac{5\pi}{21} + \frac{1}{3} \ln 2 \\
&\quad + \frac{11}{14\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{4\sqrt{2}}{7} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(3n-2)(4n-3)(6n-5)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(8n-7)(12n-11)(12n-7)} \right.
\end{aligned}$$

$$+ \frac{1}{(6n-2)(8n-3)(12n-5)(12n-1)} \Big] \\ = \frac{47\pi}{84\sqrt{3}} - \frac{2\pi}{7} + \frac{6}{7} \ln 3 - \frac{26}{21} \ln 2.$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)(12n-7)} \\ = \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)} \right. \\ \left. - \frac{3}{(6n-5)(8n-7)(12n-11)(12n-7)} \right] \\ = \frac{19\pi\sqrt{3}}{56} - \frac{2\pi\sqrt{2}}{7} - \frac{17\pi}{112} + \frac{117}{112} \ln 3 - \frac{15}{14} \ln 2 \\ + \frac{5\sqrt{3}}{28} \ln(2 + \sqrt{3}) - \frac{4\sqrt{2}}{7} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)(12n-1)} \\ = \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)} \right. \\ \left. - \frac{3}{(6n-2)(8n-3)(12n-5)(12n-1)} \right] \\ = \frac{\pi}{56\sqrt{3}} + \frac{2\pi\sqrt{2}}{7} - \frac{47\pi}{112} + \frac{117}{112} \ln 3 - \frac{29}{14} \ln 2 \\ + \frac{4\sqrt{2}}{7} \ln(\sqrt{2} + 1) - \frac{5\sqrt{3}}{28} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(4n-3)(6n-5)(6n-1)} \\ = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)(12n-7)} \right.$$

$$\begin{aligned}
& - \frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)(12n-1)} \Big] \\
& = \frac{\pi}{\sqrt{3}} - \frac{4\pi\sqrt{2}}{7} + \frac{15\pi}{56} + \ln 2 + \frac{5\sqrt{3}}{14} \ln(2 + \sqrt{3}) \\
& \quad - \frac{8\sqrt{2}}{7} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(4n-3)(6n-5)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(8n-7)(12n-11)(12n-7)} \right. \\
& \quad \left. + \frac{1}{(4n-1)(6n-2)(8n-3)(12n-5)(12n-1)} \right] \\
& = \frac{29\pi}{28\sqrt{3}} - \frac{4\pi}{7} + \frac{117}{56} \ln 3 - \frac{22}{7} \ln 2.
\end{aligned}$$

$$\begin{aligned}
(\text{三}) \quad & \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)} \\
& = -\frac{7\pi}{10\sqrt{3}} + \frac{\pi\sqrt{2}}{5} + \frac{\pi}{5} + \frac{3}{5} \ln 2 - \frac{3}{5} \ln 3 \\
& \quad + \frac{2\sqrt{2}}{5} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-4)(8n-7)(8n-5)} \\
& = -\frac{\pi\sqrt{3}}{20} + \frac{3\pi}{10\sqrt{2}} - \frac{\pi}{10} + \frac{6}{5} \ln 2 - \frac{9}{20} \ln 3 \\
& \quad - \frac{\sqrt{2}}{5} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(12n-11)}$$

$$= \frac{3\pi\sqrt{3}}{56} - \frac{3\pi}{28\sqrt{2}} + \frac{\pi}{28} + \frac{9}{56} \ln 3 - \frac{3}{28} \ln 2$$

$$+ \frac{3\sqrt{3}}{28} \ln(2 + \sqrt{3}) - \frac{\sqrt{2}}{7} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-5)(12n-11)(12n-7)}$$

$$= -\frac{3\pi\sqrt{3}}{112} - \frac{\pi}{14\sqrt{2}} + \frac{3\pi}{28} + \frac{3}{28} \ln 3 - \frac{1}{14} \ln 2$$

$$- \frac{3\sqrt{3}}{56} \ln(2 + \sqrt{3}) + \frac{1}{7\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)(8n-3)}$$

$$= \frac{\pi}{10\sqrt{3}} - \frac{\pi\sqrt{2}}{5} + \frac{\pi}{5} + \frac{9}{5} \ln 2 - \frac{3}{5} \ln 3$$

$$- \frac{2\sqrt{2}}{5} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-3)(8n-1)}$$

$$= \frac{3\pi\sqrt{3}}{20} - \frac{3\pi}{10\sqrt{2}} - \frac{\pi}{10} + \frac{3}{5} \ln 2 - \frac{9}{20} \ln 3$$

$$+ \frac{\sqrt{2}}{5} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-3)(8n-1)(12n-5)}$$

$$= \frac{3\pi\sqrt{3}}{56} + \frac{3\pi}{28\sqrt{2}} - \frac{5\pi}{28} + \frac{9}{56} \ln 3 - \frac{3}{28} \ln 2$$

$$- \frac{3\sqrt{3}}{28} \ln(2 + \sqrt{3}) + \frac{\sqrt{2}}{7} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-1)(12n-5)(12n-1)}$$

$$\begin{aligned}
&= -\frac{3\pi\sqrt{3}}{112} + \frac{\pi}{14\sqrt{2}} - \frac{\pi}{28} + \frac{3}{28} \ln 3 - \frac{1}{14} \ln 2 \\
&\quad + \frac{3\sqrt{3}}{56} \ln(2 + \sqrt{3}) - \frac{1}{7\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

由此顺次推得

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)(4n-3)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)} \right. \\
&\quad \left. - \frac{1}{(6n-2)(6n-1)(8n-3)} \right] \\
&= -\frac{4\pi}{5\sqrt{3}} + \frac{2\pi\sqrt{2}}{5} - \frac{6}{5} \ln 2 + \frac{4\sqrt{2}}{5} \ln(\sqrt{2} + 1), \\
&\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(4n-3)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)} \right. \\
&\quad \left. + \frac{1}{(6n-2)(6n-1)(8n-3)} \right] \\
&= -\frac{\pi\sqrt{3}}{5} + \frac{2\pi}{5} + \frac{12}{5} \ln 2 - \frac{6}{5} \ln 3, \\
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-1)(4n-3)(4n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-7)(8n-5)} \right. \\
&\quad \left. - \frac{1}{(6n-1)(8n-3)(8n-1)} \right] \\
&= -\frac{\pi\sqrt{3}}{5} + \frac{3\pi}{5\sqrt{2}} + \frac{3}{5} \ln 2 - \frac{2\sqrt{2}}{5} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-3)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-7)(8n-5)} \right. \\ \left. + \frac{1}{(6n-1)(8n-3)(8n-1)} \right]$$

$$= \frac{\pi\sqrt{3}}{10} - \frac{\pi}{5} + \frac{9}{5} \ln 2 - \frac{9}{10} \ln 3,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-5)(12n-11)} \right. \\ \left. - \frac{1}{(8n-3)(8n-1)(12n-5)} \right]$$

$$= -\frac{3\pi}{14\sqrt{2}} + \frac{3\pi}{14} + \frac{3\sqrt{3}}{14} \ln(2 + \sqrt{3}) \\ - \frac{2\sqrt{2}}{7} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-5)(12n-11)} \right. \\ \left. + \frac{1}{(8n-3)(8n-1)(12n-5)} \right]$$

$$= \frac{3\pi\sqrt{3}}{28} - \frac{\pi}{7} + \frac{9}{28} \ln 3 - \frac{3}{14} \ln 2,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-5)(12n-11)(12n-7)} \right]$$

$$\begin{aligned}
& - \frac{1}{(8n-1)(12n-5)(12n-1)} \Big] \\
& = -\frac{\pi}{7\sqrt{2}} + \frac{\pi}{7} - \frac{3\sqrt{3}}{28} \ln(2 + \sqrt{3}) \\
& \quad + \frac{\sqrt{2}}{7} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-5)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-5)(12n-11)(12n-7)} \right. \\
& \quad \left. + \frac{1}{(8n-1)(12n-5)(12n-1)} \right] \\
& = -\frac{3\pi\sqrt{3}}{56} + \frac{\pi}{14} + \frac{3}{14} \ln 3 - \frac{1}{7} \ln 2.
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)} \\
& = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3}{(6n-5)(6n-4)(8n-7)} \right. \\
& \quad \left. - \frac{4}{(6n-4)(8n-7)(8n-5)} \right] \\
& = -\frac{\pi\sqrt{3}}{10} + \frac{\pi}{5} - \frac{3}{5} \ln 2 + \frac{2\sqrt{2}}{5} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)} \\
& = -\frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-7)(8n-5)} \right. \\
& \quad \left. - \frac{2}{(8n-7)(8n-5)(12n-11)} \right]
\end{aligned}$$

$$= \frac{11\pi}{70\sqrt{3}} - \frac{3\pi\sqrt{2}}{35} + \frac{2\pi}{85} + \frac{9}{85} \ln 3 - \frac{33}{70} \ln 2$$

$$+ \frac{\sqrt{3}}{14} \ln(2 + \sqrt{3}) - \frac{\sqrt{2}}{35} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$= \frac{1}{7} \sum_{n=1}^{\infty} \left[ \frac{2}{(8n-7)(8n-5)(12n-11)} \right.$$

$$\left. - \frac{3}{(8n-5)(12n-11)(12n-7)} \right]$$

$$= \frac{3\pi\sqrt{3}}{112} - \frac{\pi}{28} + \frac{3\sqrt{3}}{56} \ln(2 + \sqrt{3})$$

$$- \frac{1}{7\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3}{(6n-2)(6n-1)(8n-3)} \right.$$

$$\left. - \frac{4}{(6n-1)(8n-3)(8n-1)} \right]$$

$$= -\frac{\pi\sqrt{3}}{10} + \frac{\pi}{5} + \frac{3}{5} \ln 2 - \frac{2\sqrt{2}}{5} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)}$$

$$= -\frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-1)(8n-3)(8n-1)} \right.$$

$$\left. - \frac{2}{(8n-3)(8n-1)(12n-5)} \right]$$

$$= -\frac{\pi\sqrt{3}}{70} + \frac{3\pi\sqrt{2}}{35} - \frac{3\pi}{35} - \frac{19}{70} \ln 2 + \frac{9}{35} \ln 3$$



$$+ \frac{\sqrt{2}}{35} \ln(\sqrt{2} + 1) - \frac{\sqrt{3}}{14} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$= \frac{1}{7} \sum_{n=1}^{\infty} \left[ \frac{2}{(8n-3)(8n-1)(12n-5)} - \frac{3}{(8n-1)(12n-5)(12n-1)} \right]$$

$$= \frac{3\pi\sqrt{3}}{112} - \frac{\pi}{28} + \frac{1}{7\sqrt{2}} \ln(\sqrt{2} + 1)$$

$$- \frac{3\sqrt{3}}{56} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)} - \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)} \right]$$

$$= -\frac{6}{5} \ln 2 + \frac{4\sqrt{2}}{5} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)} + \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)} \right]$$

$$= -\frac{\pi\sqrt{3}}{5} + \frac{2\pi}{5},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. - \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)} \right] \\
&= \frac{\pi}{5\sqrt{3}} - \frac{6\pi\sqrt{2}}{35} + \frac{\pi}{7} - \frac{1}{5} \ln 2 + \frac{\sqrt{3}}{7} \ln(2 + \sqrt{3}) \\
&\quad - \frac{2\sqrt{2}}{35} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-3)(4n-1)(6n-5)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. + \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)} \right] \\
&= \frac{4\pi}{35\sqrt{3}} - \frac{\pi}{35} + \frac{18}{35} \ln 3 - \frac{26}{35} \ln 2,
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\
&\quad \left. - \frac{1}{(8n-3)(8n-1)(12n-5)(12n-1)} \right] \\
&= \frac{3\sqrt{3}}{28} \ln(2 + \sqrt{3}) - \frac{\sqrt{2}}{7} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\
&\quad \left. + \frac{1}{(8n-3)(8n-1)(12n-5)(12n-1)} \right]
\end{aligned}$$

$$= \frac{3\pi\sqrt{3}}{56} - \frac{\pi}{14}.$$

$$= \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$= - \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)} \right. \\ \left. - \frac{2}{(6n-4)(8n-7)(8n-5)(12n-11)} \right]$$

$$= \frac{43\pi}{70\sqrt{3}} - \frac{6\pi\sqrt{2}}{35} - \frac{3\pi}{35} + \frac{18}{35} \ln 3 - \frac{12}{35} \ln 2 \\ + \frac{\sqrt{3}}{7} \ln(2 + \sqrt{3}) - \frac{16\sqrt{2}}{35} \ln(\sqrt{2} + 1),$$

$$= \sum_{n=1}^{\infty} \frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\ \left. - \frac{2}{(8n-7)(8n-5)(12n-11)(12n-7)} \right]$$

$$= -\frac{\pi}{280\sqrt{3}} - \frac{3\pi\sqrt{2}}{35} + \frac{9\pi}{70} + \frac{9}{35} \ln 3 - \frac{33}{70} \ln 2 \\ + \frac{4\sqrt{2}}{35} \ln(\sqrt{2} + 1) - \frac{\sqrt{3}}{28} \ln(2 + \sqrt{3}),$$

$$= \sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)}$$

$$= - \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)} \right. \\ \left. - \frac{2}{(6n-1)(8n-3)(8n-1)(12n-5)} \right]$$

$$= \frac{\pi\sqrt{3}}{14} + \frac{6\pi\sqrt{2}}{35} - \frac{13\pi}{35} + \frac{18}{35} \ln 3 - \frac{8}{7} \ln 2$$

$$+ \frac{16\sqrt{2}}{35} \ln(\sqrt{2} + 1) - \frac{\sqrt{3}}{7} \ln(2 + \sqrt{3}),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)} \right.$$

$$\left. - \frac{2}{(8n-3)(8n-1)(12n-5)(12n-1)} \right]$$

$$= -\frac{19\pi\sqrt{3}}{280} + \frac{3\pi\sqrt{2}}{35} - \frac{\pi}{70} + \frac{9}{35} \ln 3 - \frac{19}{70} \ln 2$$

$$- \frac{4\sqrt{2}}{35} \ln(\sqrt{2} + 1) + \frac{\sqrt{3}}{28} \ln(2 + \sqrt{3}).$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right.$$

$$\left. - \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \right]$$

$$= \frac{2\pi}{5\sqrt{3}} - \frac{12\pi\sqrt{2}}{35} + \frac{2\pi}{7} + \frac{4}{5} \ln 2$$

$$+ \frac{2\sqrt{3}}{7} \ln(2 + \sqrt{3}) - \frac{32\sqrt{2}}{35} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right.$$

$$\left. + \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \right]$$

$$= \frac{29\pi}{35\sqrt{3}} - \frac{16\pi}{35} + \frac{36}{35} \ln 3 - \frac{52}{35} \ln 2,$$

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\ & \quad \left. - \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right] \\ &= \frac{\pi}{5\sqrt{3}} - \frac{6\pi\sqrt{2}}{35} + \frac{\pi}{7} - \frac{1}{5} \ln 2 \\ & \quad + \frac{8\sqrt{2}}{35} \ln(\sqrt{2}+1) - \frac{2\sqrt{3}}{28} \ln(2+\sqrt{3}), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\ & \quad \left. + \frac{1}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right] \\ &= -\frac{29\pi}{140\sqrt{3}} + \frac{4\pi}{35} + \frac{18}{35} \ln 3 - \frac{26}{35} \ln 2. \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\ & \quad \left. - \frac{2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right] \\ &= \frac{29\pi}{140\sqrt{3}} - \frac{4\pi}{35} + \frac{1}{5} \ln 2 + \frac{\sqrt{3}}{14} \ln(2+\sqrt{3}) \end{aligned}$$

$$- \frac{8\sqrt{2}}{35} \ln(\sqrt{2} + 1),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \right. \\ & \quad \left. - \frac{2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right] \\ &= \frac{29\pi}{140\sqrt{3}} - \frac{4\pi}{35} - \frac{1}{5} \ln 2 + \frac{8\sqrt{2}}{35} \ln(\sqrt{2} + 1) \\ & \quad - \frac{\sqrt{3}}{14} \ln(2 + \sqrt{3}); \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\ & \quad \left. - \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right] \\ &= \frac{2}{5} \ln 2 + \frac{\sqrt{3}}{7} \ln(2 + \sqrt{3}) - \frac{16\sqrt{2}}{35} \ln(\sqrt{2} + 1), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\ & \quad \left. + \frac{1}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right] \end{aligned}$$

$$= \frac{29\pi}{70\sqrt{3}} - \frac{8\pi}{35}.$$

$$(四) \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-5)(6n-1)}$$

$$= -\frac{3\pi\sqrt{3}}{56} + \frac{\pi}{14} + \frac{3}{14} \ln 3 - \frac{1}{7} \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-1)(8n-7)}$$

$$= -\frac{9\pi\sqrt{3}}{136} + \frac{\pi\sqrt{2}}{17} + \frac{\pi}{17} + \frac{4}{17} \ln 2 - \frac{3}{17} \ln 3$$

$$+ \frac{2\sqrt{2}}{17} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-7)(8n-1)}$$

$$= \frac{3\pi\sqrt{3}}{68} - \frac{2\pi\sqrt{2}}{51} - \frac{2\pi}{51} + \frac{3}{17} \ln 2 - \frac{9}{68} \ln 3$$

$$+ \frac{3}{17\sqrt{2}} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-1)(12n-11)}$$

$$= \frac{3\pi\sqrt{3}}{152} - \frac{5\pi}{114\sqrt{2}} + \frac{\pi}{57} + \frac{9}{152} \ln 3 - \frac{3}{76} \ln 2$$

$$+ \frac{3\sqrt{3}}{76} \ln(2+\sqrt{3}) - \frac{3}{38\sqrt{2}} \ln(\sqrt{2}+1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-1)(12n-11)(12n-1)}$$

$$= -\frac{9\pi\sqrt{3}}{760} + \frac{\pi}{38\sqrt{2}} - \frac{\pi}{95} + \frac{3}{76} \ln 3 - \frac{1}{38} \ln 2$$

$$+ \frac{\sqrt{3}}{38} \ln(2 + \sqrt{3}) - \frac{1}{19\sqrt{2}} \ln(\sqrt{2} + 1).$$

由此顺次求得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)} \\ &= -\frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-5)(6n-1)} \right. \\ & \quad \left. - \frac{2}{(6n-5)(6n-1)(8n-7)} \right] \\ &= -\frac{15\pi\sqrt{3}}{952} + \frac{2\pi\sqrt{2}}{85} + \frac{11\pi}{1190} + \frac{73}{595} \ln 2 \\ & \quad - \frac{27}{238} \ln 3 + \frac{4\sqrt{2}}{85} \ln(\sqrt{2} + 1), \\ & \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)} \\ &= \frac{1}{17} \sum_{n=1}^{\infty} \left[ \frac{3}{(6n-5)(6n-1)(8n-7)} \right. \\ & \quad \left. - \frac{4}{(6n-1)(8n-7)(8n-1)} \right] \\ &= -\frac{51\pi\sqrt{3}}{2312} + \frac{\pi\sqrt{2}}{51} + \frac{\pi}{51}, \\ & \sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-7)(8n-1)(12n-11)} \\ &= -\frac{1}{9} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-1)(8n-7)(8n-1)} \right. \\ & \quad \left. - \frac{2}{(8n-7)(8n-1)(12n-11)} \right] \\ &= -\frac{\pi}{646\sqrt{3}} - \frac{\pi}{969\sqrt{2}} + \frac{8\pi}{969} - \frac{55}{1938} \ln 2 + \frac{9}{323} \ln 3 \end{aligned}$$



$$+ \frac{1}{38\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{6\sqrt{2}}{323} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-1)(12n-11)(12n-1)}$$

$$= \frac{1}{19} \sum_{n=1}^{\infty} \left[ \frac{2}{(8n-7)(8n-1)(12n-11)} \right. \\ \left. - \frac{3}{(8n-1)(12n-11)(12n-1)} \right]$$

$$= \frac{57\pi\sqrt{3}}{14440} - \frac{\pi}{114\sqrt{2}} + \frac{\pi}{285},$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)}$$

$$= - \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)} \right. \\ \left. - \frac{2}{(4n-1)(6n-5)(6n-1)(8n-7)} \right]$$

$$= - \frac{81\pi\sqrt{3}}{952} + \frac{4\pi\sqrt{2}}{85} + \frac{107\pi}{1190} + \frac{146}{595} \ln 2 \\ - \frac{27}{119} \ln 3 + \frac{8\sqrt{2}}{85} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)} \right. \\ \left. - \frac{2}{(6n-5)(6n-1)(8n-7)(8n-1)} \right]$$

$$= \frac{27\pi\sqrt{3}}{952} - \frac{4\pi\sqrt{2}}{255} - \frac{107\pi}{3570} + \frac{73}{595} \ln 2$$

$$- \frac{27}{238} \ln 3 + \frac{4\sqrt{2}}{85} \ln(\sqrt{2} + 1),$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)} \\ &= - \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)} \right. \\ & \quad \left. - \frac{2}{(6n-1)(8n-7)(8n-1)(12n-11)} \right] \\ &= \frac{163\pi}{2584\sqrt{3}} - \frac{20\pi\sqrt{2}}{969} - \frac{\pi}{323} - \frac{55}{969} \ln 2 + \frac{18}{332} \ln 3 \\ & \quad + \frac{1}{19\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{12\sqrt{2}}{323} \ln(\sqrt{2} + 1), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-7)(8n-1)(12n-11)(12n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-1)(8n-7)(8n-1)(12n-11)} \right. \\ & \quad \left. - \frac{2}{(8n-7)(8n-1)(12n-11)(12n-1)} \right] \\ &= - \frac{163\pi}{6460\sqrt{3}} + \frac{8\pi\sqrt{2}}{969} + \frac{2\pi}{1615} - \frac{55}{1938} \ln 2 + \frac{9}{323} \ln 3 \\ & \quad + \frac{1}{38\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{6\sqrt{2}}{323} \ln(\sqrt{2} + 1); \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)} \\ &= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)} \right. \\ & \quad \left. - \frac{2}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)} \right] \end{aligned}$$

$$= -\frac{27\pi\sqrt{3}}{952} + \frac{4\pi\sqrt{2}}{255} + \frac{107\pi}{3570},$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)} \\ &= -\frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-1)} \right. \\ & \quad \left. - \frac{3}{(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)} \right] \\ &= \frac{157\pi\sqrt{3}}{36176} - \frac{28\pi\sqrt{2}}{4845} + \frac{1403\pi}{542640} - \frac{414}{11305} \ln 2 \\ & \quad + \frac{1269}{36176} \ln 3 + \frac{\sqrt{3}}{152} \ln(2 + \sqrt{3}) \\ & \quad - \frac{32\sqrt{2}}{1615} \ln(\sqrt{2} + 1), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)(12n-1)} \\ &= \frac{1}{9} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-1)(8n-7)(8n-1)(12n-11)} \right. \\ & \quad \left. - \frac{2}{(6n-1)(8n-7)(8n-1)(12n-11)(12n-1)} \right] \\ &= \frac{163\pi}{12920\sqrt{3}} - \frac{4\pi\sqrt{2}}{969} - \frac{\pi}{1615}. \end{aligned}$$

$$\begin{aligned} \text{(五)} \quad & \sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-7)(8n-5)} \\ &= -\frac{3\pi\sqrt{3}}{748} - \frac{3\pi}{374\sqrt{2}} + \frac{7\pi}{374} - \frac{3}{187} \ln 2 + \frac{9}{748} \ln 3 \\ & \quad + \frac{7\sqrt{2}}{187} \ln(\sqrt{2} + 1), \end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(12n-11)} \\
&= \frac{3\pi\sqrt{3}}{56} - \frac{3\pi}{28\sqrt{2}} + \frac{\pi}{28} + \frac{9}{56} \ln 3 - \frac{3}{28} \ln 2 \\
&\quad + \frac{3\sqrt{3}}{28} \ln(2 + \sqrt{3}) - \frac{\sqrt{2}}{7} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} \frac{1}{(8n-5)(12n-11)(12n-7)} \\
&= -\frac{3\pi\sqrt{3}}{112} - \frac{\pi}{14\sqrt{2}} + \frac{3\pi}{28} + \frac{3}{28} \ln 3 - \frac{1}{14} \ln 2 \\
&\quad - \frac{3\sqrt{3}}{56} \ln(2 + \sqrt{3}) + \frac{1}{7\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

由此顺次推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3}{(6n-5)(6n-1)(8n-7)} \right. \\
&\quad \left. - \frac{4}{(6n-1)(8n-7)(8n-5)} \right] \\
&= -\frac{273\pi\sqrt{3}}{7480} + \frac{36\pi\sqrt{2}}{935} + \frac{19\pi}{935} + \frac{144}{935} \ln 2 \\
&\quad - \frac{108}{935} \ln 3 + \frac{38\sqrt{2}}{935} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-7)(8n-5)(12n-11)} \\
&= -\frac{1}{9} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-1)(8n-7)(8n-5)} \right. \\
&\quad \left. - \frac{2}{(8n-7)(8n-5)(12n-11)} \right]
\end{aligned}$$

$$= \frac{97\pi\sqrt{3}}{7854} - \frac{15\pi\sqrt{2}}{1309} + \frac{23\pi}{3927} + \frac{45}{1309} \ln 3 - \frac{173}{7854} \ln 2$$

$$+ \frac{1}{14\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{47\sqrt{2}}{1309} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$= \frac{1}{7} \sum_{n=1}^{\infty} \left[ \frac{2}{(8n-7)(8n-5)(12n-11)} - \frac{3}{(8n-5)(12n-11)(12n-7)} \right]$$

$$= \frac{3\pi\sqrt{3}}{112} - \frac{\pi}{28} + \frac{3\sqrt{3}}{56} \ln(2 + \sqrt{3})$$

$$- \frac{1}{7\sqrt{2}} \ln(\sqrt{2} + 1);$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)}$$

$$= -\frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)} - \frac{2}{(6n-5)(6n-1)(8n-7)(8n-5)} \right]$$

$$= -\frac{999\pi\sqrt{3}}{52360} + \frac{10\pi\sqrt{2}}{561} + \frac{137\pi}{13090} + \frac{1213}{19635} \ln 2$$

$$- \frac{513}{13090} \ln 3 + \frac{32\sqrt{2}}{2805} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)}$$

$$= -\sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)} \right]$$

$$\begin{aligned}
& - \frac{2}{(6n-1)(8n-7)(8n-5)(12n-11)} \Big] \\
& = \frac{9613\pi\sqrt{3}}{157080} - \frac{402\pi\sqrt{2}}{6545} - \frac{169\pi}{19635} - \frac{3889}{19635} \ln 2 \\
& + \frac{1206}{6545} \ln 3 - \frac{736\sqrt{2}}{6545} \ln(\sqrt{2}+1) \\
& + \frac{1}{7\sqrt{3}} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-1)(8n-7)(8n-5)(12n-11)(12n-7)} \\
& = \frac{37\pi\sqrt{3}}{4488} + \frac{3\pi\sqrt{2}}{1309} - \frac{607\pi}{39270} - \frac{9}{1309} \ln 3 \\
& + \frac{173}{39270} \ln 2 + \frac{1}{20\sqrt{3}} \ln(2+\sqrt{3}) \\
& - \frac{4\sqrt{2}}{187} \ln(\sqrt{2}+1);
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(4n-1)(6n-5)(6n-1)(8n-7)} \right. \\
& \quad \left. - \frac{2}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)} \right] \\
& = -\frac{351\pi\sqrt{3}}{7480} + \frac{32\pi\sqrt{2}}{2805} + \frac{129\pi}{1870} + \frac{2392}{19635} \ln 2 \\
& - \frac{972}{6545} \ln 3 + \frac{40\sqrt{2}}{561} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)}$$

$$\begin{aligned}
&= -\frac{1}{8} \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-1)(6n-5)(6n-1)(8n-7)(8n-5)} \right. \\
&\quad \left. - \frac{3}{(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)} \right] \\
&= \frac{379\pi\sqrt{3}}{14960} - \frac{496\pi\sqrt{2}}{19635} - \frac{95\pi}{20944} - \frac{46}{561} \ln 2 \\
&\quad + \frac{1107}{14960} \ln 3 - \frac{856\sqrt{2}}{19635} \ln(\sqrt{2}+1) \\
&\quad + \frac{\sqrt{3}}{56} \ln(2+\sqrt{3}),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)(12n-7)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{1}{(6n-5)(6n-1)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. - \frac{2}{(6n-1)(8n-7)(8n-5)(12n-11)(12n-7)} \right] \\
&= \frac{2341\pi\sqrt{3}}{157080} - \frac{144\pi\sqrt{2}}{6545} + \frac{146\pi}{19635} + \frac{432}{6545} \ln 3 \\
&\quad - \frac{1354}{19635} \ln 2 - \frac{152\sqrt{2}}{6545} \ln(\sqrt{2}+1) \\
&\quad + \frac{1}{70\sqrt{3}} \ln(2+\sqrt{3}).
\end{aligned}$$

$$\begin{aligned}
7. \quad \sum_{n=1}^{\infty} &\left[ \frac{A}{k(A_1 n - A_1) + a} - \frac{B}{k(B_1 n - B_1) + b} \right. \\
&\quad \left. + \frac{D}{k(D_1 n - D_1) + d} \right]
\end{aligned}$$

**定理15** 设  $A, B, b, d$  都是正整数, 则

$$\sum_{n=1}^{\infty} \left[ \frac{A}{k(2n-2)+a} - \frac{3B}{k(3n-3)+b} + \frac{4B-2A}{k(4n-4)+d} \right]$$

$$= AC(a, 2k) - 3BC(b, 3k) + (4B-2A)C(d, 4k)$$

$$+ \frac{1}{2k} \ln \frac{2^{4B-A} b^{2B}}{3^{2B} a^A d^{B-2A}}; \quad (1)$$

$$\sum_{n=1}^{\infty} \left[ \frac{A}{k(3n-3)+a} - \frac{2B}{k(4n-4)+b} + \frac{3B-2A}{k(6n-6)+d} \right]$$

$$= AC(a, 3k) - 2BC(b, 4k) + (3B-2A)C(d, 6k)$$

$$+ \frac{1}{6k} \ln \frac{3^{2A} 6^{3B-2A} b^{3B}}{4^{3B} a^{2A} d^{3B-2A}}; \quad (2)$$

$$\sum_{n=1}^{\infty} \left[ \frac{A}{k(2n-2)+a} - \frac{B}{k(3n-3)+b} + \frac{2B-3A}{k(6n-6)+d} \right]$$

$$= AC(a, 2k) - BC(b, 3k) + (2B-3A)C(d, 6k)$$

$$+ \frac{1}{6k} \ln \frac{2^{3A} 6^{2B-3A} b^{2B}}{3^{2B} a^{3A} d^{2B-3A}}; \quad (3)$$

$$\sum_{n=1}^{\infty} \left[ \frac{3A}{k(3n-3)+a} - \frac{3B}{k(6n-6)+b} + \frac{4B-8A}{k(8n-8)+d} \right]$$

$$= 3AC(a, 3k) - 3BC(b, 6k) + (4B-8A)C(d, 8k)$$

$$+ \frac{1}{2k} \ln \frac{3^{2A} 8^{B-2A} b^B}{6^B a^{2A} d^{B-2A}}; \quad (4)$$

$$\sum_{n=1}^{\infty} \left[ \frac{3A}{k(3n-3)+a} - \frac{B}{k(4n-4)+b} + \frac{2B-8A}{k(8n-8)+d} \right]$$

$$= 3AC(a, 3k) - BC(b, 4k) + (2B-8A)C(d, 8k)$$

$$+ \frac{1}{4k} \ln \frac{3^{4A} 2^{B-12A} b^B}{a^{4A} d^{B-4A}}; \quad (5)$$

$$\sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{2B}{k(2n-2)+b} + \frac{3B-3A}{k(3n-3)+d} \right]$$

$$= AC(a, k) - 2BC(b, 2k) + (3B-3A)C(d, 3k)$$



$$+ \frac{1}{k} \ln \frac{3^{B-A} b^B}{2^B a^A d^{B-A}}; \quad (6)$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{A}{k(3n-3)+a} - \frac{2B}{k(2n-2)+b} + \frac{3B-A}{k(3n-3)+d} \right] \\ &= AC(a, 3k) - 2BC(b, 2k) + (3B-A)C(d, 3k) \\ &+ \frac{1}{3k} \ln \frac{3^{3B} b^{3B}}{2^{3B} a^A d^{3B-A}}; \end{aligned} \quad (7)$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{4A}{k(4n-4)+a} - \frac{B}{k(3n-3)+b} + \frac{B-3A}{k(3n-3)+d} \right] \\ &= 4AC(a, 4k) - BC(b, 3k) + (B-3A)C(d, 3k) \\ &+ \frac{1}{3k} \ln \frac{4^{3A} 3^{B-3A} b^B}{3^B a^{3A} d^{B-3A}}; \end{aligned} \quad (8)$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{3A}{k(3n-3)+a} - \frac{B}{k(2n-2)+b} + \frac{B-2A}{k(2n-2)+d} \right] \\ &= 3AC(a, 3k) - BC(b, 2k) + (B-2A)C(d, 2k) \\ &+ \frac{1}{2k} \ln \frac{3^{2A} 2^{B-2A} b^B}{2^B a^{2A} d^{B-2A}}; \end{aligned} \quad (9)$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{A}{k(4n-4)+a} - \frac{3B}{k(3n-3)+b} + \frac{4B-A}{k(4n-4)+d} \right] \\ &= AC(a, 4k) - 3BC(b, 3k) + (4B-A)C(d, 4k) \\ &+ \frac{1}{4k} \ln \frac{4^{4B} b^{4B}}{3^{4B} a^A d^{4B-A}}. \end{aligned} \quad (10)$$

**证明** 根据定理 2, 有

$$\begin{aligned} & \sum_{i=1}^n \left[ \frac{A}{k(2i-2)+a} - \frac{3B}{k(3i-3)+b} + \frac{4B-2A}{k(4i-4)+d} \right] \\ &= AC(a, 2k) - 3BC(b, 3k) + (4B-2A)C(d, 4k) \\ &+ \frac{1}{12k} \ln \frac{(2nk-2k+a)^{3A} (4nk-4k+d)^{12B-6A}}{(3nk-3k+b)^{12B}} \end{aligned}$$

$$+ \frac{1}{12k} \ln \frac{b^{12B}}{a^{6A} d^{12B-6A}} + A\varepsilon_n(a, 2k) - 3B\varepsilon_n(b, 3k) \\ + (4B - 2A)\varepsilon_n(d, 4k).$$

令  $n \rightarrow \infty$  并取极限, 即得 (1).

仿此可证定理的其余各部分.

例1 求和:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(3n-1)}, \\ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(3n-1)}, \\ \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-2)(3n-1)}, \\ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(2n-1)(3n-2)(3n-1)}.$$

解 用分项分式方法求得

$$\frac{1}{(4n-3)(6n-5)(6n-4)} = \frac{3}{6n-5} - \frac{4}{4n-3} + \frac{3}{6n-4}.$$

于定理15(7)令  $A=3$ ,  $B=2$ ,  $a=1$ ,  $b=1$ ,  $d=2$ ,  $k=2$ ,

则有

$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(6n-4)} \\ = \sum_{n=1}^{\infty} \left( \frac{3}{6n-5} - \frac{4}{4n-3} + \frac{3}{6n-4} \right) \\ = 3C(1,6) - 4C(1,4) + \frac{3}{2} C(1,3) + \frac{1}{2} \ln \frac{3^2}{2^3}.$$

根据定理6, 有

$$3C(1,6) - 2C(1,4) = \frac{\pi\sqrt{3}}{4} - \frac{\pi}{4} + \frac{1}{4} \ln 3,$$

$$\frac{3}{2} C(1, 3) - 2C(1, 4) = \frac{\pi}{4\sqrt{3}} - \frac{\pi}{4} + \frac{1}{4} \ln \frac{3}{2^2}.$$

于是 
$$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(6n-4)}$$

$$= \frac{\pi}{\sqrt{3}} - \frac{\pi}{2} + \frac{3}{2} \ln 3 - 2 \ln 2.$$

仿此求得

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(6n-2)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left( \frac{3}{6n-2} - \frac{4}{4n-1} + \frac{3}{6n-1} \right)$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2} + \frac{3}{2} \ln 3 - 2 \ln 2.$$

用分项分式方法求得

$$\frac{n}{(4n-3)(6n-5)(6n-4)}$$

$$= \frac{1}{2} \left( \frac{5}{6n-5} - \frac{6}{4n-3} + \frac{4}{6n-4} \right).$$

于定理15(7)令  $A=5$ ,  $B=3$ ,  $a=1$ ,  $b=1$ ,  $d=2$ ,  $k=2$ , 则有

$$\sum_{n=1}^{\infty} \frac{n}{(4n-3)(6n-5)(6n-4)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{5}{6n-5} - \frac{6}{4n-3} + \frac{4}{6n-4} \right)$$

$$= \frac{5}{2} C(1, 6) - 3C(1, 4) + C(1, 3) + \frac{1}{12} \ln \frac{3^3}{2^{12}}.$$

根据定理6, 有

$$\frac{5}{2} C(1, 6) - \frac{5}{3} C(1, 4)$$

$$= \frac{5\pi}{8\sqrt{3}} - \frac{5\pi}{24} + \frac{1}{24} \ln 3^5,$$

$$C(1,3) - \frac{4}{3} C(1,4)$$

$$= \frac{\pi}{6\sqrt{3}} - \frac{\pi}{6} + \frac{1}{6} \ln \frac{3}{2}.$$

于是 
$$\sum_{n=1}^{\infty} \frac{n}{(4n-3)(6n-5)(6n-4)}$$

$$= \frac{19\pi}{24\sqrt{3}} - \frac{3\pi}{8} + \frac{9}{8} \ln 3 - \frac{17}{12} \ln 2.$$

仿此求得

$$\sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-2)(6n-1)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{2}{6n-2} - \frac{2}{4n-1} + \frac{1}{6n-1} \right)$$

$$= -\frac{5\pi}{24\sqrt{3}} + \frac{\pi}{8} + \frac{3}{8} \ln 3 - \frac{7}{12} \ln 2.$$

从而 
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-2)(3n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)} + \frac{1}{(4n-1)(6n-2)(6n-1)} \right]$$

$$= 3 \ln 3 - 4 \ln 2,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(3n-2)(3n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(4n-3)(6n-5)(6n-4)} \right.$$

$$\left. - \frac{1}{(4n-1)(6n-2)(6n-1)} \right]$$

$$= \frac{2\pi}{\sqrt{3}} - \pi,$$

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-2)(3n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(4n-3)(6n-5)(6n-4)} + \frac{2n}{(4n-1)(6n-2)(6n-1)} \right]$$

$$= 2 \sum_{n=1}^{\infty} \frac{n}{(4n-3)(6n-5)(6n-4)}$$

$$- \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(6n-4)}$$

$$+ 2 \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-2)(6n-1)}$$

$$= \frac{\pi}{6\sqrt{3}} + \frac{3}{2} \ln 3 - 2 \ln 2,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(2n-1)(3n-2)(3n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(4n-3)(6n-5)(6n-4)} - \frac{2n}{(4n-1)(6n-2)(6n-1)} \right]$$

$$= \frac{\pi}{\sqrt{3}} - \frac{\pi}{2} + \frac{1}{3} \ln 2.$$

例2 求和:

$$\sum_{n=1}^{\infty} \frac{n}{(3n-2)(4n-1)(6n-1)(8n+1)}.$$

解 用分项分式方法求得

$$\frac{n}{(3n-2)(4n-1)(6n-1)}$$

$$= \frac{1}{15} \left( \frac{2}{3n-2} - \frac{6}{4n-1} + \frac{5}{6n-1} \right).$$

于定理15(2)令  $A=2$ ,  $B=3$ ,  $a=1$ ,  $b=3$ ,  $d=5$ ,  $k=1$ , 则有

$$\sum_{n=1}^{\infty} \frac{n}{(3n-2)(4n-1)(6n-1)}$$

$$= \frac{1}{15} \sum_{n=1}^{\infty} \left( \frac{2}{3n-2} - \frac{6}{4n-1} + \frac{5}{6n-1} \right)$$

$$= \frac{2}{15} C(1,3) - \frac{2}{5} C(3,4) + \frac{1}{3} C(5,6)$$

$$+ \frac{1}{90} \ln \frac{3^{18}}{2^{18} 5^6}.$$

根据定理6, 有

$$\frac{2}{15} C(1,3) - \frac{8}{45} C(3,4)$$

$$= \frac{\pi}{45\sqrt{3}} + \frac{\pi}{45} - \frac{1}{45} \ln 2^2 \cdot 3,$$

$$\frac{1}{3} C(5,6) - \frac{2}{9} C(3,4)$$

$$= -\frac{\pi}{12\sqrt{3}} + \frac{\pi}{36} + \frac{1}{36} \ln \frac{5^2}{3}.$$

于是

$$\sum_{n=1}^{\infty} \frac{n}{(3n-2)(4n-6)(6n-1)}$$

$$= -\frac{11\pi}{180\sqrt{3}} + \frac{\pi}{20} + \frac{3}{20} \ln 3 - \frac{17}{90} \ln 2.$$

用分项分式方法求得

$$\begin{aligned} & \frac{n}{(4n-1)(6n-1)(8n+1)} \\ &= \frac{1}{42} \left( \frac{7}{4n-1} - \frac{9}{6n-1} - \frac{2}{8n+1} \right). \end{aligned}$$

于定理 15(1) 令  $A=7$ ,  $B=3$ ,  $a=3$ ,  $b=5$ ,  $d=9$ ,  $k=2$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-1)(8n+1)} \\ &= \frac{1}{42} \sum_{n=1}^{\infty} \left( \frac{7}{4n-1} - \frac{9}{6n-1} - \frac{2}{8n+1} \right) \\ &= \frac{1}{6} C(3,4) - \frac{3}{14} C(5,6) - \frac{1}{21} C(9,8) \\ & \quad + \frac{1}{168} \ln \frac{2^5 5^6}{3^{11}}. \end{aligned}$$

于命题 2 令  $a=1$ ,  $b=1$ ,  $k=8$ , 则有

$$\begin{aligned} C(9,8) &= C(1,8) + \frac{1}{8} \ln 3^2 - 1, \\ -\frac{1}{21} C(9,8) &= -\frac{1}{21} C(1,8) - \frac{1}{168} \ln 3^2 + \frac{1}{21}. \end{aligned}$$

$$\begin{aligned} \text{于是} \quad & \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-1)(8n+1)} \\ &= \frac{1}{6} C(3,4) - \frac{3}{14} C(5,6) - \frac{1}{21} C(1,8) \\ & \quad + \frac{1}{168} \ln \frac{2^5 5^6}{3^{13}} + \frac{1}{21}. \end{aligned}$$

根据定理 6, 有

$$\frac{1}{7} C(3,4) - \frac{3}{14} C(5,6)$$

$$= \frac{\pi\sqrt{3}}{56} - \frac{\pi}{56} + \frac{1}{56} \ln \frac{3}{5^2},$$

$$\frac{1}{42} C(3,4) - \frac{1}{21} C(1,8)$$

$$= -\frac{\pi}{168\sqrt{2}} - \frac{\pi}{168} + \frac{1}{168} \ln 3 \\ - \frac{1}{84\sqrt{2}} \ln(\sqrt{2} + 1).$$

于是 
$$\sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-1)(8n+1)}$$

$$= \frac{\pi\sqrt{3}}{56} - \frac{\pi}{168\sqrt{2}} - \frac{\pi}{42} + \frac{5}{168} \ln 2 - \frac{3}{56} \ln 3 \\ - \frac{1}{84\sqrt{2}} \ln(\sqrt{2} + 1) + \frac{1}{21}.$$

从而 
$$\sum_{n=1}^{\infty} \frac{n}{(3n-2)(4n-1)(6n-1)(8n+1)}$$

$$= \frac{1}{19} \sum_{n=1}^{\infty} \left[ \frac{3n}{(3n-2)(4n-1)(6n-1)} \right. \\ \left. - \frac{8n}{(4n-1)(6n-1)(8n+1)} \right]$$

$$= -\frac{257\pi\sqrt{3}}{23940} + \frac{\pi}{399\sqrt{2}} + \frac{143\pi}{7980} + \frac{123}{2660} \ln 3 \\ - \frac{169}{3990} \ln 2 + \frac{\sqrt{2}}{399} \ln(\sqrt{2} + 1) - \frac{8}{399}.$$

例3 求和: .

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

解 用分项分式方法求得



$$\frac{n}{(2n-1)(3n-1)(4n-1)} = \frac{1}{2n-1} - \frac{3}{3n-1} + \frac{2}{4n-1}.$$

于定理15(1)令  $A=1$ ,  $B=1$ ,  $a=1$ ,  $b=2$ ,  $d=3$ ,  $k=1$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)} \\ &= \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{3}{3n-1} + \frac{2}{4n-1} \right) \\ &= C(1,2) - 3C(2,3) + 2C(3,4) + \frac{1}{2} \ln \frac{2^5}{3^3}. \end{aligned}$$

根据定理6, 有

$$\begin{aligned} C(1,2) - \frac{3}{2} C(2,3) &= \frac{\pi}{4\sqrt{3}} - \frac{1}{4} \ln 3, \\ 2C(3,4) - \frac{3}{2} C(2,3) &= \frac{\pi}{4\sqrt{3}} - \frac{\pi}{4} + \frac{1}{4} \ln 3. \end{aligned}$$

$$\begin{aligned} \text{于是 } \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)} \\ &= \frac{\pi}{2\sqrt{3}} - \frac{\pi}{4} + \frac{5}{2} \ln 2 - \frac{3}{2} \ln 3. \end{aligned}$$

于定理15(2)令  $A=1$ ,  $B=1$ ,  $a=2$ ,  $b=3$ ,  $d=5$ ,  $k=1$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(6n-1)} \\ &= \sum_{n=1}^{\infty} \left( \frac{1}{3n-1} - \frac{3}{4n-1} + \frac{2}{6n-1} \right) \\ &= -\frac{5\pi}{12\sqrt{3}} + \frac{\pi}{4} + \frac{3}{4} \ln 3 - \frac{7}{6} \ln 2, \end{aligned}$$

于定理15(1)令  $A=1$ ,  $B=1$ ,  $a=3$ ,  $b=5$ ,  $d=7$ ,  $k=2$ ,

即可推得

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-1)(8n-1)} \\
 &= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{4n-1} - \frac{3}{6n-1} + \frac{2}{8n-1} \right) \\
 &= \frac{\pi\sqrt{3}}{8} - \frac{\pi}{8\sqrt{2}} - \frac{\pi}{8} + \frac{3}{8} \ln 2 - \frac{3}{8} \ln 3 \\
 &\quad + \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1);
 \end{aligned}$$

于定理 15(2) 令  $A=1$ ,  $B=1$ ,  $a=5$ ,  $b=7$ ,  $d=11$ ,  $k=2$ , 即可推得

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{n}{(6n-1)(8n-1)(12n-1)} \\
 &= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{6n-1} - \frac{2}{8n-1} + \frac{1}{12n-1} \right) \\
 &= -\frac{\pi\sqrt{3}}{16} + \frac{\pi}{8\sqrt{2}} + \frac{\pi}{48} + \frac{3}{16} \ln 3 - \frac{5}{24} \ln 2 \\
 &\quad + \frac{1}{8\sqrt{3}} \ln(2+\sqrt{3}) - \frac{1}{4\sqrt{2}} \ln(\sqrt{2}+1).
 \end{aligned}$$

由此顺次求得

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)} \\
 &= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(4n-1)} \right. \\
 &\quad \left. - \frac{3n}{(3n-1)(4n-1)(6n-1)} \right] \\
 &= \frac{7\pi}{8\sqrt{3}} - \frac{\pi}{2} + 3 \ln 2 - \frac{15}{8} \ln 3,
 \end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(6n-1)(8n-1)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3n}{(3n-1)(4n-1)(6n-1)} \right. \\
&\quad \left. - \frac{8n}{(4n-1)(6n-1)(8n-1)} \right] \\
&= -\frac{17\pi}{20\sqrt{3}} + \frac{\pi}{5\sqrt{2}} + \frac{7\pi}{20} + \frac{21}{20} \ln 3 - \frac{13}{10} \ln 2 \\
&\quad - \frac{\sqrt{2}}{5} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-1)(8n-1)(12n-1)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{n}{(4n-1)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{3n}{(6n-1)(8n-1)(12n-1)} \right] \\
&= \frac{5\pi\sqrt{3}}{32} - \frac{\pi}{4\sqrt{2}} - \frac{3\pi}{32} + \frac{1}{2} \ln 2 - \frac{15}{32} \ln 3 \\
&\quad - \frac{\sqrt{3}}{16} \ln(2+\sqrt{3}) + \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)} \right. \\
&\quad \left. - \frac{4n}{(3n-1)(4n-1)(6n-1)(8n-1)} \right] \\
&= \frac{19\pi\sqrt{3}}{40} - \frac{2\pi\sqrt{2}}{15} - \frac{19\pi}{30} + \frac{41}{15} \ln 2 - \frac{81}{40} \ln 3 \\
&\quad + \frac{4\sqrt{2}}{15} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(3n-1)(4n-1)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{4n}{(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\
&= -\frac{109\pi}{120\sqrt{3}} + \frac{\pi\sqrt{2}}{5} + \frac{29\pi}{120} + \frac{39}{40} \ln 3 - \frac{11}{10} \ln 2 \\
&\quad - \frac{2\sqrt{2}}{5} \ln(\sqrt{2}+1) + \frac{1}{4\sqrt{3}} \ln(2+\sqrt{3}).
\end{aligned}$$

从而

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{6n}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\
&= \frac{11\pi}{8\sqrt{3}} - \frac{4\pi\sqrt{2}}{15} - \frac{5\pi}{12} + \frac{28}{15} \ln 2 - \frac{63}{40} \ln 3 \\
&\quad + \frac{8\sqrt{2}}{15} \ln(\sqrt{2}+1) - \frac{\sqrt{3}}{10} \ln(2+\sqrt{3}).
\end{aligned}$$

**例4** 求和:

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

**解** 于定理15(6)令  $A=1, B=2, a=1, b=3, d=5,$   
 $k=2$ , 即可推得

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)(4n-1)(6n-1)}$$

$$\begin{aligned}
&= \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{4}{4n-1} + \frac{3}{6n-1} \right) \\
&= -\frac{\pi\sqrt{3}}{16} + \frac{\pi}{8} + \frac{3}{16} \ln 3 - \frac{1}{4} \ln 2.
\end{aligned}$$

由此以及例 3 中的相应结果即可顺次推得

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n}{(2n-1)(4n-1)(6n-1)(8n-1)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(4n-1)(6n-1)} \right. \\
&\quad \left. - \frac{4n}{(4n-1)(6n-1)(8n-1)} \right] \\
&= -\frac{3\pi\sqrt{3}}{16} + \frac{\pi}{6\sqrt{2}} + \frac{5\pi}{24} + \frac{9}{16} \ln 3 - \frac{7}{12} \ln 2 \\
&\quad - \frac{1}{3\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

从而

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(4n-1)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{6n}{(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\
&= -\frac{9\pi\sqrt{3}}{40} + \frac{\pi}{3\sqrt{2}} + \frac{37\pi}{240} + \frac{27}{40} \ln 3 - \frac{43}{60} \ln 2 \\
&\quad + \frac{3\sqrt{3}}{40} \ln(2+\sqrt{3}) - \frac{\sqrt{2}}{3} \ln(\sqrt{2}+1).
\end{aligned}$$

**例5** 求和:

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)}.$$

**解** 于定理 15(3) 令  $A=1$ ,  $B=2$ ,  $a=1$ ,  $b=2$ ,  $d$

$=5, k=1$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(6n-1)} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{2}{3n-1} + \frac{1}{6n-1} \right) \\ &= \frac{\pi}{24\sqrt{3}} + \frac{2}{3} \ln 2 - \frac{3}{8} \ln 3, \end{aligned}$$

于定理15(4)令  $A=1, B=5, a=2, b=5, d=7, k=1$ .  
即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(3n-1)(6n-1)(8n-1)} \\ &= \frac{1}{15} \sum_{n=1}^{\infty} \left( \frac{3}{3n-1} - \frac{15}{6n-1} + \frac{12}{8n-1} \right) \\ &= \frac{13\pi}{60\sqrt{3}} - \frac{\pi}{10\sqrt{2}} - \frac{\pi}{20} + \frac{1}{15} \ln 2 - \frac{3}{20} \ln 3 \\ & \quad + \frac{1}{5\sqrt{2}} \ln(\sqrt{2}+1). \end{aligned}$$

于是

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{4n}{(2n-1)(3n-1)(6n-1)(8n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(6n-1)} \right. \\ & \quad \left. - \frac{4n}{(3n-1)(6n-1)(8n-1)} \right] \\ &= -\frac{11\pi}{40\sqrt{3}} + \frac{\pi\sqrt{2}}{15} + \frac{\pi}{15} + \frac{2}{15} \ln 2 + \frac{3}{40} \ln 3 \\ & \quad - \frac{2\sqrt{2}}{15} \ln(\sqrt{2}+1), \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{n}{(3n-1)(6n-1)(8n-1)(12n-1)}$$

$$\begin{aligned}
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(3n-1)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{4n}{(6n-1)(8n-1)(12n-1)} \right] \\
&= \frac{29\pi}{90\sqrt{3}} - \frac{\pi}{5\sqrt{2}} - \frac{2\pi}{45} + \frac{3}{10} \ln 2 - \frac{3}{10} \ln 3 \\
&\quad + \frac{\sqrt{2}}{5} \ln(\sqrt{2}+1) - \frac{1}{6\sqrt{3}} \ln(2+\sqrt{3}).
\end{aligned}$$

从而

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(6n-1)(8n-1)} \right. \\
&\quad \left. - \frac{6n}{(3n-1)(6n-1)(8n-1)(12n-1)} \right] \\
&= -\frac{53\pi}{120\sqrt{3}} + \frac{2\pi\sqrt{2}}{15} + \frac{\pi}{15} - \frac{1}{3} \ln 2 + \frac{3}{8} \ln 3 \\
&\quad - \frac{4\sqrt{2}}{15} \ln(\sqrt{2}+1) + \frac{1}{5\sqrt{3}} \ln(2+\sqrt{3}).
\end{aligned}$$

**例6** 求和:

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)}.$$

**解** 于定理 15(5) 令  $A=1$ ,  $B=5$ ,  $a=2$ ,  $b=3$ ,  $d=7$ ,  $k=1$ , 即可推得

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(8n-1)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left( \frac{3}{3n-1} - \frac{5}{4n-1} + \frac{2}{8n-1} \right) \\
&= -\frac{\pi}{10\sqrt{3}} - \frac{\pi}{20\sqrt{2}} + \frac{\pi}{10} + \frac{3}{10} \ln 3 - \frac{11}{20} \ln 2
\end{aligned}$$

$$+ \frac{1}{10\sqrt{2}} \ln(\sqrt{2} + 1),$$

于定理15(6)令  $A=1$ ,  $B=2$ ,  $a=3$ ,  $b=7$ ,  $d=11$ ,  $k=4$ ,  
即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(4n-1)(8n-1)(12n-1)} \\ &= \frac{1}{8} \sum_{n=1}^{\infty} \left( \frac{1}{4n-1} - \frac{4}{8n-1} + \frac{3}{12n-1} \right) \\ &= -\frac{\pi\sqrt{3}}{64} + \frac{\pi}{16\sqrt{2}} - \frac{\pi}{64} + \frac{3}{64} \ln 3 - \frac{1}{16} \ln 2 \\ & \quad + \frac{\sqrt{3}}{32} \ln(2 + \sqrt{3}) - \frac{1}{8\sqrt{2}} \ln(\sqrt{2} + 1). \end{aligned}$$

于是

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(8n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(4n-1)} \right. \\ & \quad \left. - \frac{4n}{(3n-1)(4n-1)(8n-1)} \right] \\ &= \frac{\pi\sqrt{3}}{10} + \frac{\pi}{15\sqrt{2}} - \frac{13\pi}{60} + \frac{47}{30} \ln 2 - \frac{9}{10} \ln 3 \\ & \quad - \frac{\sqrt{2}}{15} \ln(\sqrt{2} + 1), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(8n-1)(12n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(3n-1)(4n-1)(8n-1)} \right. \\ & \quad \left. - \frac{4n}{(4n-1)(8n-1)(12n-1)} \right] \\ &= \frac{7\pi}{240\sqrt{3}} - \frac{\pi}{10\sqrt{2}} + \frac{13\pi}{240} + \frac{3}{80} \ln 3 - \frac{1}{10} \ln 2. \end{aligned}$$



$$+ \frac{1}{5\sqrt{2}} \ln(\sqrt{2} + 1) - \frac{1}{8\sqrt{3}} \ln(2 + \sqrt{3}).$$

从而

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)} \\ &= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(4n-1)(8n-1)} \right. \\ & \quad \left. - \frac{6n}{(3n-1)(4n-1)(8n-1)(12n-1)} \right] \\ &= \frac{\pi}{40\sqrt{3}} + \frac{\pi\sqrt{2}}{15} - \frac{13\pi}{120} + \frac{13}{30} \ln 2 - \frac{9}{40} \ln 3 \\ & \quad - \frac{2\sqrt{2}}{15} \ln(\sqrt{2} + 1) + \frac{\sqrt{3}}{20} \ln(2 + \sqrt{3}). \end{aligned}$$

例7 求和:

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)}.$$

解 于定理15(3)令  $A=1, B=2, a=3, b=5, d=11,$   
 $k=2$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-1)(12n-1)} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{1}{4n-1} - \frac{2}{6n-1} + \frac{1}{12n-1} \right) \\ &= \frac{\pi\sqrt{3}}{32} - \frac{5\pi}{96} + \frac{1}{12} \ln 2 - \frac{3}{32} \ln 3 \\ & \quad + \frac{1}{16\sqrt{3}} \ln(2 + \sqrt{3}). \end{aligned}$$

于是

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(6n-1)(12n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(3n-1)(4n-1)(6n-1)} \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{4n}{(4n-1)(6n-1)(12n-1)} \Big] \\
& = -\frac{19\pi}{72\sqrt{3}} + \frac{11\pi}{72} + \frac{3}{8} \ln 3 - \frac{1}{2} \ln 2 \\
& \quad - \frac{1}{12\sqrt{3}} \ln(2 + \sqrt{3}),
\end{aligned}$$

从而

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)} \\
& = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)} \right. \\
& \quad \left. - \frac{6n}{(3n-1)(4n-1)(6n-1)(12n-1)} \right] \\
& = \frac{59\pi}{120\sqrt{3}} - \frac{17\pi}{60} + \frac{6}{5} \ln 2 - \frac{33}{40} \ln 3 \\
& \quad + \frac{1}{10\sqrt{3}} \ln(2 + \sqrt{3}).
\end{aligned}$$

**例8** 求和:

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(2n-1)(3n-1)(4n-1)}, \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(3n-1)(4n-1)(6n-1)}, \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)}.
\end{aligned}$$

**解** 于定理15(1)令  $A=3, B=4, a=1, b=2, d=2,$   
 $k=2$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(4n-3)(6n-4)(8n-5)} \\
& = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{3}{4n-3} - \frac{12}{6n-4} + \frac{10}{8n-5} \right)
\end{aligned}$$

$$= -\frac{\pi}{2\sqrt{3}} + \frac{5\pi}{8\sqrt{2}} - \frac{\pi}{8} + \frac{29}{8} \ln 2 - \frac{3}{2} \ln 2 \\ - \frac{5}{4\sqrt{2}} \ln(\sqrt{2} + 1),$$

于定理15(2)令  $A=4$ ,  $B=5$ ,  $a=2$ ,  $b=5$ ,  $d=5$ ,  $k=2$ ,  
即可推得

$$\sum_{n=1}^{\infty} \frac{n}{(6n-4)(8n-5)(12n-7)} \\ = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{4}{6n-4} - \frac{10}{8n-5} + \frac{7}{12n-7} \right) \\ = -\frac{13\pi}{48\sqrt{3}} - \frac{5\pi}{8\sqrt{2}} + \frac{29\pi}{48} + \frac{15}{16} \ln 3 - \frac{13}{8} \ln 2 \\ - \frac{7}{8\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{5}{4\sqrt{2}} \ln(\sqrt{2} + 1).$$

于是

$$\sum_{n=1}^{\infty} \frac{n}{(4n-3)(6n-4)(8n-5)(12n-7)} \\ = \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{n}{(4n-3)(6n-4)(8n-5)} \right. \\ \left. - \frac{3n}{(6n-4)(8n-5)(12n-7)} \right] \\ = \frac{5\pi}{32\sqrt{3}} + \frac{5\pi}{4\sqrt{2}} - \frac{31\pi}{32} + \frac{17}{4} \ln 2 - \frac{69}{32} \ln 3 \\ + \frac{7\sqrt{3}}{16} \ln(2 + \sqrt{3}) - \frac{5}{2\sqrt{2}} \ln(\sqrt{2} + 1).$$

由此以及上面已经求得的相应的结果, 即可推得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(2n-1)(3n-1)(4n-1)} \\ = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(4n-3)(6n-4)(8n-5)} \right]$$

$$\begin{aligned}
& - \frac{2n}{(4n-1)(6n-1)(8n-1)} \Big] \\
& = 2 \sum_{n=1}^{\infty} \frac{n}{(4n-3)(6n-4)(8n-5)} \\
& \quad - \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-4)(8n-5)} \\
& \quad - 2 \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-1)(8n-1)} \\
& = -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2\sqrt{2}} + \frac{\pi}{4} + \ln 2 - \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(3n-1)(4n-1)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-4)(8n-5)(12n-7)} \right. \\
& \quad \left. - \frac{2n}{(6n-1)(8n-1)(12n-1)} \right] \\
& = \frac{\pi}{3\sqrt{3}} - \frac{\pi}{2\sqrt{2}} + \frac{\pi}{6} - \frac{1}{3} \ln 2 + \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1) \\
& \quad - \frac{1}{2\sqrt{3}} \ln(2+\sqrt{3}), \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(4n-3)(6n-4)(8n-5)(12n-7)} \right. \\
& \quad \left. - \frac{2n}{(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\
& = 2 \sum_{n=1}^{\infty} \frac{n}{(4n-3)(6n-4)(8n-5)(12n-7)} \\
& \quad - \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-4)(8n-5)(12n-7)}
\end{aligned}$$

$$\begin{aligned}
& - 2 \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-1)(8n-1)(12n-1)} \\
& = -\frac{\pi}{\sqrt{3}} + \frac{\pi}{\sqrt{2}} - \frac{\pi}{8} + \ln 2 + \frac{\sqrt{3}}{4} \ln(2 + \sqrt{3}) \\
& \quad - \sqrt{2} \ln(\sqrt{2} + 1).
\end{aligned}$$

仿此求得以下一系列的结果。

于定理15(8)令  $A=7$ ,  $B=25$ ,  $a=1$ ,  $b=1$ ,  $d=2$ ,  $k=2$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(8n-7)} \\
& = \frac{1}{10} \sum_{n=1}^{\infty} \left( \frac{28}{8n-7} - \frac{25}{6n-5} + \frac{4}{6n-4} \right) \\
& = -\frac{71\pi}{120\sqrt{3}} + \frac{7\pi}{20\sqrt{2}} + \frac{7\pi}{40} + \frac{17}{30} \ln 2 - \frac{21}{40} \ln 3 \\
& \quad + \frac{7}{10\sqrt{2}} \ln(\sqrt{2} + 1);
\end{aligned}$$

于定理 15(10) 令  $A=7$ ,  $B=8$ ,  $a=1$ ,  $b=2$ ,  $d=3$ ,  $k=2$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-4)(8n-7)(8n-5)} \\
& = \frac{1}{20} \sum_{n=1}^{\infty} \left( \frac{4}{8n-7} - \frac{24}{6n-4} + \frac{25}{8n-5} \right) \\
& = -\frac{\pi}{10\sqrt{3}} + \frac{\pi}{5\sqrt{2}} - \frac{9\pi}{160} + \frac{4}{5} \ln 2 - \frac{3}{10} \ln 3 \\
& \quad - \frac{9}{40\sqrt{2}} \ln(\sqrt{2} + 1);
\end{aligned}$$

于定理15(9)令  $A=22$ ,  $B=49$ ,  $a=1$ ,  $b=1$ ,  $d=3$ ,  $k=4$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-5)(12n-11)} \\
&= \frac{1}{56} \sum_{n=1}^{\infty} \left( \frac{66}{12n-11} - \frac{49}{8n-7} + \frac{5}{8n-5} \right) \\
&= \frac{11\pi\sqrt{3}}{224} - \frac{11\pi}{112\sqrt{2}} + \frac{17\pi}{448} + \frac{33}{224} \ln 3 - \frac{11}{112} \ln 2 \\
&\quad + \frac{11\sqrt{3}}{112} \ln(2 + \sqrt{3}) - \frac{27}{112\sqrt{2}} \ln(\sqrt{2} + 1);
\end{aligned}$$

于定理15(7)令  $A=11$ ,  $B=20$ ,  $a=1$ ,  $b=3$ ,  $d=5$ ,  $k=4$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-5)(12n-11)(12n-7)} \\
&= \frac{1}{112} \sum_{n=1}^{\infty} \left( \frac{11}{12n-11} - \frac{40}{8n-5} + \frac{49}{12n-7} \right) \\
&= -\frac{19\pi}{448\sqrt{3}} - \frac{5\pi}{112\sqrt{2}} + \frac{15\pi}{224} + \frac{15}{224} \ln 3 - \frac{5}{112} \ln 2 \\
&\quad - \frac{19}{224\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{5}{56\sqrt{2}} \ln(\sqrt{2} + 1);
\end{aligned}$$

于定理15(8)令  $A=3$ ,  $B=10$ ,  $a=5$ ,  $b=4$ ,  $d=5$ ,  $k=2$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-2)(6n-1)(8n-3)} \\
&= \frac{1}{10} \sum_{n=1}^{\infty} \left( \frac{12}{8n-3} - \frac{10}{6n-2} + \frac{1}{6n-1} \right) \\
&= \frac{7\pi}{120\sqrt{3}} - \frac{3\pi}{20\sqrt{2}} + \frac{3\pi}{40} + \frac{19}{30} \ln 2 - \frac{9}{40} \ln 3 \\
&\quad - \frac{3}{10\sqrt{2}} \ln(\sqrt{2} + 1);
\end{aligned}$$

于定理 15(10) 令  $A=3, B=2, a=5, b=5, d=7,$   
 $k=2,$  即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(6n-1)(8n-3)(8n-1)} \\ &= \frac{1}{20} \sum_{n=1}^{\infty} \left( \frac{3}{8n-3} - \frac{6}{6n-1} + \frac{5}{8n-1} \right) \\ &= \frac{\pi\sqrt{3}}{40} - \frac{\pi}{20\sqrt{2}} - \frac{\pi}{160} + \frac{1}{10} \ln 2 - \frac{3}{40} \ln 3 \\ & \quad + \frac{1}{40\sqrt{2}} \ln(\sqrt{2}+1); \end{aligned}$$

于定理 15(9) 令  $A=10, B=21, a=7, b=5, d=7,$   
 $k=4,$  即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(8n-3)(8n-1)(12n-5)} \\ &= \frac{1}{56} \sum_{n=1}^{\infty} \left( \frac{30}{12n-5} - \frac{21}{8n-3} + \frac{1}{8n-1} \right) \\ &= \frac{5\pi\sqrt{3}}{224} + \frac{5\pi}{112\sqrt{2}} - \frac{31\pi}{448} + \frac{15}{224} \ln 3 - \frac{5}{112} \ln 2 \\ & \quad - \frac{5\sqrt{3}}{112} \ln(2+\sqrt{3}) + \frac{11}{112\sqrt{2}} \ln(\sqrt{2}+1); \end{aligned}$$

于定理 15(7) 令  $A=5, B=4, a=7, b=7, d=11,$   
 $k=4,$  即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(8n-1)(12n-5)(12n-1)} \\ &= \frac{1}{112} \sum_{n=1}^{\infty} \left( \frac{5}{12n-5} - \frac{8}{8n-1} + \frac{7}{12n-1} \right) \\ &= -\frac{\pi}{448\sqrt{3}} + \frac{\pi}{112\sqrt{2}} - \frac{\pi}{224} + \frac{3}{224} \ln 3 - \frac{1}{112} \ln 2 \end{aligned}$$

$$+ \frac{1}{224\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{56\sqrt{2}} \ln(\sqrt{2} + 1).$$

由此以及上面已求得的相应结果即可顺次推得

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(3n-2)(3n-1)(4n-3)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-5)(6n-4)(8n-7)} \right. \\ & \quad \left. - \frac{2n}{(6n-2)(6n-1)(8n-3)} \right] \\ &= 2 \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(8n-7)} \\ & \quad - \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)} \\ & \quad - 2 \sum_{n=1}^{\infty} \frac{n}{(6n-2)(6n-1)(8n-3)} \\ &= -\frac{\pi\sqrt{3}}{5} + \frac{3\pi}{5\sqrt{2}} - \frac{11}{15} \ln 2 + \frac{3\sqrt{2}}{5} \ln(\sqrt{2} + 1), \\ & \sum_{n=1}^{\infty} \frac{n}{(3n-2)(3n-1)(4n-3)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-5)(6n-4)(8n-7)} \right. \\ & \quad \left. + \frac{2n}{(6n-2)(6n-1)(8n-3)} \right] \\ &= -\frac{11\pi}{30\sqrt{3}} + \frac{3\pi}{10} + \frac{9}{5} \ln 2 - \frac{9}{10} \ln 3, \\ & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(3n-1)(4n-3)(4n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-4)(8n-7)(8n-5)} \right. \end{aligned}$$



$$\begin{aligned}
& - \frac{2n}{(6n-1)(8n-3)(8n-1)} \Big] \\
& = -\frac{\pi}{5\sqrt{3}} + \frac{\pi}{5\sqrt{2}} + \frac{1}{5} \ln 2 - \frac{1}{10\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-3)(4n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-4)(8n-7)(8n-5)} \right. \\
& \quad \left. + \frac{2n}{(6n-1)(8n-3)(8n-1)} \right] \\
& = \frac{\pi}{10\sqrt{3}} - \frac{\pi}{40} + \frac{3}{5} \ln 2 - \frac{3}{10} \ln 3, \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(4n-3)(4n-1)(6n-5)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-7)(8n-5)(12n-11)} \right. \\
& \quad \left. - \frac{2n}{(8n-3)(8n-1)(12n-5)} \right] \\
& = -\frac{5\pi}{28\sqrt{2}} + \frac{5\pi}{28} + \frac{5\sqrt{3}}{28} \ln(2+\sqrt{3}) \\
& \quad - \frac{11}{28\sqrt{2}} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-1)(6n-5)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-7)(8n-5)(12n-11)} \right. \\
& \quad \left. + \frac{2n}{(8n-3)(8n-1)(12n-5)} \right] \\
& = \frac{5\pi\sqrt{3}}{56} - \frac{11\pi}{112} + \frac{15}{56} \ln 3 - \frac{5}{28} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(4n-1)(6n-5)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-5)(12n-11)(12n-7)} \right. \\
&\quad \left. - \frac{2n}{(8n-1)(12n-5)(12n-1)} \right] \\
&= -\frac{\pi\sqrt{2}}{56} + \frac{\pi}{28} - \frac{1}{56\sqrt{3}} \ln(2+\sqrt{3}) \\
&\quad + \frac{1}{14\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-5)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-5)(12n-11)(12n-7)} \right. \\
&\quad \left. + \frac{2n}{(8n-1)(12n-5)(12n-1)} \right] \\
&= -\frac{\pi}{112\sqrt{3}} + \frac{\pi}{56} + \frac{3}{56} \ln 3 - \frac{1}{28} \ln 2.
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3n}{(6n-5)(6n-4)(8n-7)} \right. \\
&\quad \left. - \frac{4n}{(6n-4)(8n-7)(8n-5)} \right] \\
&= -\frac{11\pi}{40\sqrt{3}} + \frac{\pi}{20\sqrt{2}} + \frac{3\pi}{20} - \frac{3}{10} \ln 2 - \frac{3}{40} \ln 3 \\
&\quad + \frac{3}{5\sqrt{2}} \ln(\sqrt{2}+1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-4)(8n-7)(8n-5)(12n-11)} \\
&= -\frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-4)(8n-7)(8n-5)} \right. \\
&\quad \left. - \frac{2n}{(8n-7)(8n-5)(12n-11)} \right] \\
&= \frac{221\pi}{1680\sqrt{3}} - \frac{37\pi}{280\sqrt{2}} + \frac{37\pi}{840} + \frac{111}{560} \ln 3 - \frac{93}{280} \ln 2 \\
&\quad + \frac{11}{56\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{3}{35\sqrt{2}} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-7)(8n-5)(12n-11)(12n-7)} \\
&= \frac{1}{7} \sum_{n=1}^{\infty} \left[ \frac{2n}{(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. - \frac{3n}{(8n-5)(12n-11)(12n-7)} \right] \\
&= \frac{9\pi\sqrt{3}}{448} - \frac{\pi}{112\sqrt{2}} - \frac{\pi}{56} + \frac{3}{224} \ln 3 - \frac{1}{112} \ln 2 \\
&\quad + \frac{9\sqrt{3}}{224} \ln(2 + \sqrt{3}) - \frac{3}{28\sqrt{2}} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{3n}{(6n-2)(6n-1)(8n-3)} \right. \\
&\quad \left. - \frac{4n}{(6n-1)(8n-3)(8n-1)} \right] \\
&= -\frac{\pi}{40\sqrt{3}} - \frac{\pi}{20\sqrt{2}} + \frac{\pi}{20} + \frac{3}{10} \ln 2 - \frac{3}{40} \ln 3 \\
&\quad - \frac{1}{5\sqrt{2}} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-1)(8n-3)(8n-1)(12n-5)} \\
&= -\frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-1)(8n-3)(8n-1)} \right. \\
&\quad \left. - \frac{2n}{(8n-3)(8n-1)(12n-5)} \right] \\
&= \frac{11\pi}{560\sqrt{3}} + \frac{13\pi}{280\sqrt{2}} - \frac{37\pi}{840} + \frac{39}{560} \ln 3 - \frac{53}{840} \ln 2 \\
&\quad - \frac{5}{56\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{\sqrt{2}}{35} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(8n-3)(8n-1)(12n-5)(12n-1)} \\
&= \frac{1}{7} \sum_{n=1}^{\infty} \left[ \frac{2n}{(8n-3)(8n-1)(12n-5)} \right. \\
&\quad \left. - \frac{3n}{(8n-1)(12n-5)(12n-1)} \right] \\
&= \frac{3\pi\sqrt{3}}{448} + \frac{\pi}{112\sqrt{2}} - \frac{\pi}{56} + \frac{3}{224} \ln 3 - \frac{1}{112} \ln 2 \\
&\quad - \frac{3\sqrt{3}}{224} \ln(2 + \sqrt{3}) + \frac{1}{28\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)} \right. \\
&\quad \left. - \frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)} \right] \\
&= 2 \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)} \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)}
\end{aligned}$$

$$\begin{aligned}
& - 2 \sum_{n=1}^{\infty} \frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)} \\
& = -\frac{\pi}{5\sqrt{3}} + \frac{\pi}{5\sqrt{2}} - \frac{3}{5} \ln 2 + \frac{2\sqrt{2}}{5} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)} \right. \\
& \quad \left. + \frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)} \right] \\
& = -\frac{\pi\sqrt{3}}{10} + \frac{\pi}{5} + \frac{3}{5} \ln 2 - \frac{3}{10} \ln 3, \\
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(3n-1)(4n-3)(4n-1)(6n-5)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
& \quad \left. - \frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)} \right] \\
& = \frac{\pi}{15\sqrt{3}} - \frac{13\pi}{70\sqrt{2}} + \frac{5\pi}{42} - \frac{1}{15} \ln 2 \\
& \quad + \frac{5}{14\sqrt{8}} \ln(2+\sqrt{3}) - \frac{4\sqrt{2}}{35} \ln(\sqrt{2}+1), \\
& \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-3)(4n-1)(6n-5)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
& \quad \left. + \frac{2n}{(6n-1)(8n-3)(8n-1)(12n-11)} \right] \\
& = \frac{61\pi}{420\sqrt{3}} - \frac{2\pi}{35} + \frac{39}{140} \ln 3 - \frac{67}{210} \ln 2,
\end{aligned}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\ \left. - \frac{2n}{(8n-3)(8n-1)(12n-5)(12n-1)} \right]$$

$$= -\frac{\pi}{28\sqrt{2}} + \frac{\pi}{28} + \frac{3\sqrt{3}}{56} \ln(2 + \sqrt{3})$$

$$- \frac{1}{7\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\ \left. + \frac{2n}{(8n-3)(8n-1)(12n-5)(12n-1)} \right]$$

$$= \frac{3\pi\sqrt{3}}{112} - \frac{\pi}{28} + \frac{3}{56} \ln 3 - \frac{1}{28} \ln 2.$$

$$\sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$= - \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)} \right. \\ \left. - \frac{2n}{(6n-4)(8n-7)(8n-5)(12n-11)} \right]$$

$$= \frac{113\pi}{210\sqrt{3}} - \frac{11\pi}{35\sqrt{2}} - \frac{13\pi}{210} + \frac{33}{70} \ln 3 - \frac{51}{140} \ln 2$$

$$+ \frac{11}{28\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{27}{35\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. - \frac{2n}{(8n-7)(8n-5)(12n-11)(12n-7)} \right] \\
&= \frac{37\pi}{3360\sqrt{3}} - \frac{2\pi\sqrt{2}}{35} + \frac{67\pi}{840} + \frac{6}{35} \ln 3 - \frac{11}{35} \ln 2 \\
&\quad - \frac{5}{112\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{9}{70\sqrt{2}} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} \frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \\
&= - \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)} \right. \\
&\quad \left. - \frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)} \right] \\
&= \frac{3\pi\sqrt{3}}{140} + \frac{\pi}{7\sqrt{2}} - \frac{29\pi}{210} + \frac{3}{14} \ln 3 - \frac{179}{420} \ln 2 \\
&\quad - \frac{5}{28\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{11}{35\sqrt{2}} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} \frac{n}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-1)(8n-3)(8n-1)(12n-5)} \right. \\
&\quad \left. - \frac{2n}{(8n-3)(8n-1)(12n-5)(12n-1)} \right] \\
&= -\frac{23\pi}{1120\sqrt{3}} + \frac{\pi}{35\sqrt{2}} - \frac{\pi}{120} + \frac{3}{70} \ln 3 - \frac{19}{420} \ln 2 \\
&\quad - \frac{1}{112\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{70\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. - \frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \right] \\
&= 2 \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \\
&\quad - 2 \sum_{n=1}^{\infty} \frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \\
&= \frac{\pi}{3\sqrt{3}} - \frac{2\pi\sqrt{2}}{7} + \frac{5\pi}{21} + \frac{7}{15} \ln 2 \\
&\quad + \frac{5}{7\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{22\sqrt{2}}{35} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. + \frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \right] \\
&= \frac{62\pi}{105\sqrt{3}} - \frac{11\pi}{35} + \frac{6}{7} \ln 3 - \frac{26}{21} \ln 2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\
&\quad \left. - \frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right]
\end{aligned}$$



$$\begin{aligned}
&= \frac{\pi}{15\sqrt{3}} - \frac{2\pi\sqrt{2}}{35} + \frac{\pi}{21} - \frac{1}{15} \ln 2 \\
&\quad + \frac{1}{28\sqrt{3}} \ln(2 + \sqrt{3}) + \frac{\sqrt{2}}{35} \ln(\sqrt{2} + 1), \\
&\sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\
&\quad \left. + \frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right] \\
&= -\frac{13\pi}{840\sqrt{3}} + \frac{\pi}{70} + \frac{6}{35} \ln 3 - \frac{26}{105} \ln 2.
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. - \frac{2n}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right] \\
&= \frac{289\pi}{1680\sqrt{3}} - \frac{\pi}{35\sqrt{2}} - \frac{31\pi}{420} + \frac{3}{70} \ln 3 + \frac{37}{420} \ln 2 \\
&\quad + \frac{3\sqrt{3}}{56} \ln(2 + \sqrt{3}) - \frac{6\sqrt{2}}{35} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{2n}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \Big] \\
& = \frac{59\pi}{1680\sqrt{3}} + \frac{\pi}{35\sqrt{2}} - \frac{17\pi}{420} + \frac{3}{70} \ln 3 - \frac{47}{420} \ln 2 \\
& \quad - \frac{\sqrt{3}}{56} \ln(2 + \sqrt{3}) + \frac{2\sqrt{2}}{35} \ln(\sqrt{2} + 1). \\
& \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} = \\
& = \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\
& \quad \left. - \frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right] \\
& = 2 \sum_{n=1}^{\infty} \frac{n}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \\
& \quad - \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \\
& \quad - 2 \sum_{n=1}^{\infty} \frac{n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \\
& = \frac{\pi}{15\sqrt{3}} - \frac{2\pi\sqrt{2}}{35} + \frac{\pi}{21} + \frac{1}{5} \ln 2 + \frac{\sqrt{3}}{14} \ln(2 + \sqrt{3}) \\
& \quad - \frac{8\sqrt{2}}{35} \ln(\sqrt{2} + 1), \\
& \sum_{n=1}^{\infty} \frac{n}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left[ \frac{2n-1}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\
&\quad \left. + \frac{2n}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right] \\
&= \frac{87\pi}{420\sqrt{3}} - \frac{4\pi}{35} + \frac{6}{35} \ln 3 - \frac{26}{105} \ln 2.
\end{aligned}$$

$$\begin{aligned}
8. \quad \sum_{n=1}^{\infty} &\left[ \frac{A}{k(A_1 n - A_1) + a} - \frac{B}{k(B_1 n - B_1) + b} \right. \\
&\quad \left. + \frac{D}{k(D_1 n - D_1) + d} - \frac{E}{k(E_1 n - E_1) + e} \right]
\end{aligned}$$

定理16 设  $A$ 、 $B$ 、 $D$ 、 $b$ 、 $d$ 、 $e$  都是正整数, 则

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[ \frac{A}{k(2n-2) + a} - \frac{B}{k(3n-3) + b} + \frac{2D}{k(4n-4) + d} \right. \\
&\quad \left. - \frac{3A+3D-2B}{k(6n-6) + e} \right] \\
&= AC(a, 2k) - BC(b, 3k) + 2DC(d, 4k) \\
&\quad - (3A+3D-2B)C(e, 6k) \\
&\quad + \frac{1}{6k} \ln \frac{2^3 A 4^3 D b^{2B} e^{3A+3D-2B}}{3^{2B} 6^{3A+3D-2B} a^{3A} d^{3D}} \quad (1)
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[ \frac{3A}{k(3n-3) + a} - \frac{B}{k(4n-4) + b} + \frac{3D}{k(6n-6) + d} \right. \\
&\quad \left. - \frac{8A+4D-2B}{k(8n-8) + e} \right] \\
&= 3AC(a, 3k) - BC(b, 4k) + 3DC(d, 6k) \\
&\quad - (8A+4D-2B)C(e, 8k)
\end{aligned}$$

$$+ \frac{1}{4k} \ln \frac{3^4 A 6^3 D b^B e^{4A+2D-B}}{4^B 8^{4A+2D-B} a^{4A} d^{2D}}; \quad (2)$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{A}{k(n-1)+a} - \frac{B}{k(2n-2)+b} + \frac{3D}{k(3n-3)+d} \right. \\ & \quad \left. - \frac{4A+4D-2B}{k(4n-4)+e} \right] \\ &= AC(a, k) - BC(b, 2k) + 3DC(d, 3k) \\ & \quad - (4A+4D-2B)C(e, 4k) \\ & + \frac{1}{2k} \ln \frac{3^{2D} b^B e^{2A+2D-B}}{2^B 4^{2A+2D-B} a^{2A} d^{2D}}; \quad (3) \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{A}{k(2n-2)+a} - \frac{3B}{k(3n-3)+b} + \frac{3D}{k(6n-6)+d} \right. \\ & \quad \left. - \frac{4A+4D-8B}{k(8n-8)+e} \right] \\ &= AC(a, 2k) - 3BC(b, 3k) + 3DC(d, 6k) \\ & \quad - (4A+4D-8B)C(e, 8k) \\ & + \frac{1}{2k} \ln \frac{2^A 6^D b^{2B} e^{A+D-2B}}{3^{2B} 8^{A+D-2B} a^A d^D}; \quad (4) \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{A}{k(3n-3)+a} - \frac{B}{k(6n-6)+b} + \frac{2D}{k(8n-8)+d} \right. \\ & \quad \left. - \frac{4A+3B-2B}{k(12n-12)+e} \right] \\ &= AC(a, 3k) - BC(b, 6k) + 2DC(d, 8k) \\ & \quad - (4A+3D-2B)C(e, 12k) \\ & + \frac{1}{12k} \ln \frac{3^4 A 8^3 D b^{2B} e^{4A+3D-2B}}{6^{2B} 12^{4A+3D-2B} a^{4A} d^{3D}}; \quad (5) \end{aligned}$$

$$\sum_{n=1}^{\infty} \left[ \frac{A}{k(2n-2)+a} - \frac{3B}{k(3n-3)+b} + \frac{D}{k(4n-4)+d} \right]$$

$$\begin{aligned}
& - \frac{4A + 2D - 8B}{k(8n - 8) + e} \Big] \\
& = AC(a, 2k) - 3BC(b, 3k) + DC(d, 4k) \\
& \quad - (4A + 2D - 8B)C(e, 8k) \\
& \quad + \frac{1}{4k} \ln \frac{2^{2A} 4^D b^{4B} e^{2A+D-4B}}{3^{4B} 8^{2A+D-4B} a^{2A} d^D}; \quad (6)
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{A}{k(3n-3)+a} - \frac{B}{k(4n-4)+b} + \frac{2D}{k(8n-8)+d} \right. \\
& \quad \left. - \frac{4A+3D-3B}{k(12n-12)+e} \right] \\
& = AC(a, 3k) - BC(b, 4k) + 2DC(d, 8k) \\
& \quad - (4A + 3D - 3B)C(e, 12k) \\
& \quad + \frac{1}{12k} \ln \frac{3^{4A} 8^{3D} b^{3B} e^{4A+3D-3B}}{4^{3B} 12^{4A+3D-3B} a^{4A} d^{3D}}; \quad (7)
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{A}{k(3n-3)+a} - \frac{B}{k(4n-4)+b} + \frac{D}{k(6n-6)+d} \right. \\
& \quad \left. - \frac{4A+2D-3B}{k(12n-12)+e} \right] \\
& = AC(a, 3k) - BC(b, 4k) + DC(d, 6k) \\
& \quad - (4A + 2D - 3B)C(e, 12k) \\
& \quad + \frac{1}{12k} \ln \frac{3^{4A} 6^{2D} b^{3B} e^{4A+2D-3B}}{4^{3B} 12^{4A+2D-3B} a^{4A} d^{2D}}; \quad (8)
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{A}{k(3n-3)+a} - \frac{2B}{k(2n-2)+b} + \frac{2D}{k(2n-2)+d} \right. \\
& \quad \left. - \frac{A+3D-3B}{k(3n-3)+e} \right] \\
& = AC(a, 3k) - 2BC(b, 2k) + 2DC(d, 2k) \\
& \quad - (A + 3D - 3B)C(e, 3k).
\end{aligned}$$

$$+ \frac{1}{3k} \ln \frac{3^A 2^{3D} b^{3B} e^{A+3D-3B}}{2^{3B} 3^{A+3D-3B} a^A d^{3D}}, \quad (9)$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{A}{k(4n-4)+a} + \frac{3B}{k(3n-3)+b} + \frac{3D}{k(3n-3)+d} \right. \\ & \quad \left. - \frac{A+4D-4B}{k(4n-4)+e} \right] \\ &= AC(a, 4k) - 3BC(b, 3k) + 3DC(d, 3k) \\ & \quad - (A+4D-4B)C(e, 4k) \\ & \quad + \frac{1}{4k} \ln \frac{4^A 3^{4D} b^{4B} e^{A+4D-4B}}{3^{4B} 4^{A+4D-4B} a^A d^{4D}}, \quad (10) \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{3A}{k(3n-3)+a} - \frac{B}{k(2n-2)+b} + \frac{3D}{k(3n-3)+d} \right. \\ & \quad \left. - \frac{4A+4D-2B}{k(4n-4)+e} \right] \\ &= 3AC(a, 3k) - BC(b, 2k) + 3DC(d, 3k) \\ & \quad - (4A+4D-2B)C(e, 4k) \\ & \quad + \frac{1}{2k} \ln \frac{3^{2A+2D} b^B e^{2A+2D-B}}{2^{4A+4D-B} a^{2A} d^{2D}}, \quad (11) \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{3A}{k(6n-6)+a} - \frac{B}{k(4n-4)+b} + \frac{3D}{k(3n-3)+d} \right. \\ & \quad \left. - \frac{2A+4D-B}{k(4n-4)+e} \right] \\ &= 3AC(a, 6k) - BC(b, 4k) + 3DC(d, 3k) \\ & \quad - (2A+4D-B)C(e, 4k) \\ & \quad + \frac{1}{4k} \ln \frac{6^{2A} 3^{4D} b^B e^{2A+4D-B}}{4^{2A+4D} a^{2A} d^{4D}}. \quad (12) \end{aligned}$$

**证明** 根据定理 2, 有

$$\begin{aligned}
& \sum_{i=1}^n \left[ \frac{A}{k(2i-2)+a} - \frac{B}{k(3i-3)+d} + \frac{3D}{k(4i-4)+b} \right. \\
& \quad \left. - \frac{3A+3D-2B}{k(6i-6)+e} \right] \\
&= AC(a, 2k) - BC(b, 3k) + 2DC(d, 4k) \\
& \quad - (3A+3D-2B)C(e, 6k) \\
& \quad + \frac{1}{12k} \ln \frac{(2nk-2k+a)^{6A}(4nk-4k+d)^{6D}}{(3nk-3k+b)^{4B}(6nk-6k+e)^{6A+6D-4B}} \\
& \quad + \frac{1}{12k} \ln \frac{b^{4B}e^{6A+6D-4B}}{a^{6A}d^{6D}} + A\epsilon_n(a, 2k) - B\epsilon_n(b, 3k) \\
& \quad + 2D\epsilon_n(d, 4k) - (3A+3D-2B)\epsilon_n(e, 6k).
\end{aligned}$$

令  $n \rightarrow \infty$  并取极限, 即得 (1).

仿此可证定理的其余各部分.

**例1** 求和:

$$\sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)}.$$

**解** 用分项分式方法求得

$$\begin{aligned}
& \frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)} \\
&= \frac{1}{1540} \left( \frac{165}{6n-1} - \frac{154}{4n-1} + \frac{30}{4n-3} + \frac{21}{6n+1} \right).
\end{aligned}$$

于定理 16(9) 令  $A=165$ ,  $B=77$ ,  $D=15$ ,  $a=5$ ,  $b=3$ ,  $d=1$ ,  $e=7$ ,  $k=2$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)} \\
&= \frac{1}{1540} \sum_{n=1}^{\infty} \left( \frac{165}{6n-1} - \frac{154}{4n-1} + \frac{30}{4n-3} + \frac{21}{6n+1} \right)
\end{aligned}$$

$$= \frac{3}{28} C(5,6) - \frac{1}{10} C(3,4) + \frac{3}{154} C(1,4) + \frac{3}{220} C(7,6) \\ + \frac{1}{9240} \ln \frac{3^{417}}{2^{186} 5^{165} 7^{21}}.$$

于命题 2 令  $a=1$ ,  $b=1$ ,  $k=6$ , 则有

$$C(7,6) = C(1,6) + \frac{1}{6} \ln 7 - 1,$$

$$\frac{3}{220} C(7,6) = \frac{3}{220} C(1,6) + \frac{1}{1320} \ln 7^3 - \frac{3}{220}.$$

于是 
$$\sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)}$$

$$= \frac{3}{28} C(5,6) - \frac{1}{10} C(3,4) + \frac{3}{154} C(1,4) + \frac{3}{220} C(1,6) \\ + \frac{1}{9240} \ln \frac{3^{417}}{2^{186} 5^{165}} - \frac{3}{220}.$$

根据定理 6, 有

$$\frac{3}{28} C(5,6) - \frac{1}{14} C(3,4) \\ = -\frac{\pi\sqrt{3}}{112} + \frac{\pi}{112} + \frac{1}{112} \ln \frac{5^2}{3},$$

$$\frac{3}{154} C(1,4) - \frac{3}{154} C(3,4)$$

$$= \frac{3\pi}{616} - \frac{1}{616} \ln 3^3,$$

$$\frac{3}{220} C(1,6) - \frac{1}{110} C(3,4)$$

$$= \frac{\pi\sqrt{3}}{880} + \frac{\pi}{880} - \frac{1}{880} \ln 3.$$



$$\begin{aligned}
 \text{于是 } \sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-1)(6n-1)(6n+1)} \\
 = -\frac{3\pi\sqrt{3}}{385} + \frac{23\pi}{1540} + \frac{93}{3080} \ln 3 - \frac{31}{1540} \ln 2 \\
 - \frac{3}{220}.
 \end{aligned}$$

例2 求和:

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)(8n+1)}$$

解 用分项分式方法求得

$$\begin{aligned}
 & \frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)} \\
 &= \frac{1}{20} \left( \frac{5}{2n-1} - \frac{20}{3n-1} + \frac{16}{4n-1} + \frac{1}{6n+1} \right)
 \end{aligned}$$

于定理16(1) 令  $A=5, B=20, D=8, a=1, b=2, d=3, e=7, k=1$ , 则有

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)} \\
 &= \frac{1}{20} \sum_{n=1}^{\infty} \left( \frac{5}{2n-1} - \frac{20}{3n-1} + \frac{16}{4n-1} + \frac{1}{6n+1} \right) \\
 &= \frac{1}{4} C(1,2) - C(2,3) + \frac{4}{5} C(3,4) + \frac{1}{20} C(7,6) \\
 & \quad + \frac{1}{120} \ln \frac{2^{104}}{3^{63} \cdot 7}.
 \end{aligned}$$

于命题2 令  $a=1, b=1, k=6$ , 则有

$$C(7,6) = C(1,6) + \frac{1}{6} \ln 7 - 1,$$

$$\frac{1}{20} C(7,6) = \frac{1}{20} C(1,6) + \frac{1}{120} \ln 7 - \frac{1}{20}.$$

$$\begin{aligned}
\text{于是} \quad & \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)} \\
&= \frac{1}{4} C(1,2) - C(2,3) + \frac{4}{5} C(3,4) + \frac{1}{20} C(1,6) \\
&\quad + \frac{1}{120} \ln \frac{2^{104}}{3^{63}} + \frac{1}{20}.
\end{aligned}$$

根据定理 6, 有

$$\begin{aligned}
\frac{1}{4} C(1,2) - \frac{3}{8} C(2,3) &= \frac{\pi}{16\sqrt{3}} - \frac{1}{16} \ln 3, \\
\frac{4}{5} C(3,4) - \frac{3}{5} C(2,3) &= \frac{\pi}{10\sqrt{3}} - \frac{\pi}{10} + \frac{1}{10} \ln 3, \\
\frac{1}{20} C(1,6) - \frac{1}{40} C(2,3) &= \frac{\pi}{60\sqrt{3}}.
\end{aligned}$$

$$\begin{aligned}
\text{于是} \quad & \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)} \\
&= \frac{43\pi}{240\sqrt{3}} - \frac{\pi}{10} + \frac{13}{15} \ln 2 - \frac{39}{80} \ln 3 - \frac{1}{20}.
\end{aligned}$$

用分项分式方法求得

$$\begin{aligned}
& \frac{n}{(3n-1)(4n-1)(6n+1)(8n+1)} \\
&= \frac{1}{165} \left( \frac{15}{3n-1} - \frac{22}{4n-1} + \frac{33}{6n+1} - \frac{40}{8n+1} \right).
\end{aligned}$$

于定理 16(2) 令  $A=5$ ,  $B=22$ ,  $D=11$ ,  $a=2$ ,  $b=3$ ,  $d=7$ ,  $e=9$ ,  $k=1$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(6n+1)(8n+1)} \\
&= \frac{1}{165} \sum_{n=1}^{\infty} \left( \frac{15}{3n-1} - \frac{22}{4n-1} + \frac{33}{6n+1} - \frac{40}{8n+1} \right)
\end{aligned}$$

$$= \frac{1}{11} C(2, 3) - \frac{2}{15} C(3, 4) + \frac{1}{5} C(7, 6) - \frac{8}{33} C(9, 8) \\ + \frac{1}{660} \ln \frac{3^{104}}{2^{102} 7^{22}}.$$

于命题 2 令  $a=1$ ,  $b=1$ ,  $k=6$ , 则有

$$C(7, 6) = C(1, 6) + \frac{1}{6} \ln 7 - 1,$$

$$\frac{1}{5} C(7, 6) = \frac{1}{5} C(1, 6) + \frac{1}{30} \ln 7 - \frac{1}{5}.$$

于命题 2 令  $a=1$ ,  $b=1$ ,  $k=8$ , 则有

$$C(9, 8) = C(1, 8) + \frac{1}{8} \ln 3^2 - 1,$$

$$-\frac{8}{33} C(9, 8) = -\frac{8}{33} C(1, 8) - \frac{1}{33} \ln 3^2 + \frac{8}{33}.$$

于是  $\sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(6n+1)(8n+1)}$

$$= \frac{1}{11} C(2, 3) - \frac{2}{15} C(3, 4) + \frac{1}{5} C(1, 6) - \frac{8}{33} C(1, 8) \\ + \frac{1}{660} \ln \frac{3^{64}}{2^{102}} + \frac{7}{165}.$$

根据定理 6, 有

$$\frac{1}{11} C(2, 3) - \frac{4}{33} C(3, 4) = -\frac{\pi}{66\sqrt{3}} + \frac{\pi}{66} - \frac{1}{66} \ln 3,$$

$$\frac{1}{55} C(1, 6) - \frac{2}{165} C(3, 4)$$

$$= \frac{\pi}{220\sqrt{3}} + \frac{\pi}{660} - \frac{1}{660} \ln 3,$$

$$\frac{2}{11} C(1, 6) - \frac{8}{33} C(1, 8)$$

$$= \frac{\pi}{22\sqrt{3}} - \frac{\pi}{33\sqrt{2}} - \frac{\pi}{66} + \frac{1}{66} \ln 3 - \frac{\sqrt{2}}{33} \ln(\sqrt{2} + 1).$$

于是 
$$\sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(6n+1)(8n+1)}$$

$$= \frac{23\pi}{660\sqrt{3}} - \frac{\pi}{33\sqrt{2}} + \frac{\pi}{660} + \frac{21}{220} \ln 3 - \frac{17}{110} \ln 2$$

$$- \frac{\sqrt{2}}{33} \ln(\sqrt{2} + 1) + \frac{7}{165}.$$

所以 
$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)(8n+1)}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(4n-1)(6n+1)} \right.$$

$$\left. - \frac{4n}{(3n-1)(4n-1)(6n+1)(8n+1)} \right]$$

$$= \frac{7\pi}{880\sqrt{3}} + \frac{2\pi\sqrt{2}}{165} - \frac{7\pi}{330} + \frac{49}{165} \ln 2 - \frac{153}{880} \ln 3$$

$$+ \frac{4\sqrt{2}}{165} \ln(\sqrt{2} + 1) - \frac{29}{660}.$$

**例3 求和:**

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

**解** 用分项分式方法求得

$$\frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$= \frac{1}{4} \left( \frac{1}{2n-1} - \frac{4}{3n-1} + \frac{4}{4n-1} - \frac{1}{6n-1} \right).$$

于定理16(1)令  $A=1$ ,  $B=4$ ,  $D=2$ ,  $a=1$ ,  $b=2$ ,  $d=3$ ,  
 $e=5$ ,  $k=1$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)} \\
&= \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{4}{3n-1} + \frac{4}{4n-1} - \frac{1}{6n-1} \right) \\
&= \frac{1}{4} C(1,2) - C(2,3) + C(3,4) - \frac{1}{4} C(5,6) \\
&\quad + \frac{1}{24} \ln \frac{2^{22} 5}{3^{18}}.
\end{aligned}$$

根据定理 6, 有

$$\begin{aligned}
\frac{1}{4} C(1,2) - \frac{3}{8} C(2,3) &= \frac{\pi}{16\sqrt{3}} - \frac{1}{16} \ln 3, \\
\frac{5}{6} C(3,4) - \frac{5}{8} C(2,3) &= \frac{5\pi}{48\sqrt{3}} - \frac{5\pi}{48} + \frac{1}{48} \ln 3^5, \\
\frac{1}{6} C(3,4) - \frac{1}{4} C(5,6) &= \frac{\pi}{16\sqrt{3}} - \frac{\pi}{48} + \frac{1}{48} \ln \frac{3}{5^2}.
\end{aligned}$$

于是 
$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)} \\
&= \frac{11\pi}{48\sqrt{3}} - \frac{\pi}{8} + \frac{11}{12} \ln 2 - \frac{9}{16} \ln 3.
\end{aligned}$$

于定理 16(2) 令  $A=2$ ,  $B=15$ ,  $D=5$ ,  $a=2$ ,  $b=3$ ,  $d=5$ ,  $e=7$ ,  $k=1$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(3n-1)(4n-1)(6n-1)(8n-1)} \\
&= \frac{1}{30} \sum_{n=1}^{\infty} \left( \frac{6}{3n-1} - \frac{15}{4n-1} + \frac{15}{6n-1} - \frac{6}{8n-1} \right) \\
&= -\frac{19\pi}{120\sqrt{3}} + \frac{\pi}{40\sqrt{2}} + \frac{3\pi}{40} + \frac{9}{40} \ln 3 - \frac{37}{120} \ln 2 \\
&\quad - \frac{1}{20\sqrt{2}} \ln(\sqrt{2}+1).
\end{aligned}$$

于定理16(1)令  $A=1$ ,  $B=4$ ,  $D=2$ ,  $a=3$ ,  $b=5$ ,  $d=7$ ,  
 $e=11$ ,  $k=2$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(4n-1)(6n-1)(8n-1)(12n-1)} \\ &= \frac{1}{16} \sum_{n=1}^{\infty} \left( \frac{1}{4n-1} - \frac{4}{6n-1} + \frac{4}{8n-1} - \frac{1}{12n-1} \right) \\ &= \frac{3\pi\sqrt{3}}{128} - \frac{\pi}{32\sqrt{2}} - \frac{7\pi}{384} + \frac{7}{96} \ln 2 - \frac{9}{128} \ln 3 \\ & \quad + \frac{1}{16\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{64\sqrt{3}} \ln(2+\sqrt{3}). \end{aligned}$$

由此顺次推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)} \right. \\ & \quad \left. - \frac{4n^2}{(3n-1)(4n-1)(6n-1)(8n-1)} \right] \\ &= \frac{23\pi}{80\sqrt{3}} - \frac{\pi}{30\sqrt{2}} - \frac{17\pi}{120} + \frac{43}{60} \ln 2 - \frac{39}{80} \ln 3 \\ & \quad + \frac{1}{15\sqrt{2}} \ln(\sqrt{2}+1), \\ & \sum_{n=1}^{\infty} \frac{n^2}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(3n-1)(4n-1)(6n-1)(8n-1)} \right. \\ & \quad \left. - \frac{4n^2}{(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\ &= -\frac{211\pi}{1440\sqrt{3}} + \frac{\pi}{20\sqrt{2}} + \frac{71\pi}{1440} + \frac{27}{160} \ln 3 - \frac{1}{5} \ln 2 \end{aligned}$$

$$+ \frac{1}{48\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{16\sqrt{2}} \ln(\sqrt{2} + 1),$$

于是

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \\ &= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-1)} \right. \\ & \quad \left. - \frac{6n^2}{(3n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\ &= \frac{7\pi}{30\sqrt{3}} - \frac{\pi}{15\sqrt{2}} - \frac{7\pi}{80} + \frac{23}{60} \ln 2 - \frac{3}{10} \ln 3 \\ & \quad + \frac{\sqrt{2}}{15} \ln(\sqrt{2} + 1) - \frac{1}{40\sqrt{3}} \ln(2 + \sqrt{3}). \end{aligned}$$

例4 求和:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}.$$

解 于定理 16(1) 令  $A=9$ ,  $B=64$ ,  $D=50$ ,  $a=1$ ,  $b=2$ ,  $d=3$ ,  $e=5$ ,  $k=2$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(4n-3)(6n-4)(8n-5)(12n-7)} \\ &= \frac{1}{16} \sum_{n=1}^{\infty} \left( \frac{9}{4n-3} - \frac{64}{6n-4} + \frac{100}{8n-5} - \frac{49}{12n-7} \right) \\ &= \frac{19\pi}{384\sqrt{3}} + \frac{25\pi}{32\sqrt{2}} - \frac{221\pi}{384} + \frac{89}{32} \ln 2 - \frac{177}{128} \ln 3 \\ & \quad + \frac{49}{64\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{25}{16\sqrt{2}} \ln(\sqrt{2} + 1). \end{aligned}$$

由此以及上面已求出的相应的结果即可推得

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(4n-3)(6n-4)(8n-5)(12n-7)} \right. \\
&\quad \left. - \frac{4n^2}{(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\
&= 4 \sum_{n=1}^{\infty} \frac{n^2}{(4n-3)(6n-4)(8n-5)(12n-7)} \\
&\quad - 4 \sum_{n=1}^{\infty} \frac{n}{(4n-3)(6n-4)(8n-5)(12n-7)} \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-4)(8n-5)(12n-7)} \\
&\quad - 4 \sum_{n=1}^{\infty} \frac{n^2}{(4n-1)(6n-1)(8n-1)(12n-1)} \\
&= -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4\sqrt{2}} + \frac{\pi}{48} + \frac{1}{3} \ln 2 \\
&\quad + \frac{1}{8\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

例5 求和:

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(2n-1)(3n-2)(3n-1)(4n-1)}, \\
&\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-2)(3n-1)(4n-1)}.
\end{aligned}$$

解 于定理 16 (11) 令  $A=5$ ,  $B=27$ ,  $D=16$ ,  $a=1$ ,  
 $b=1$ ,  $d=2$ ,  $e=3$ ,  $k=2$ , 即可推得

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^2}{(4n-3)(6n-5)(6n-4)(8n-5)} \\
&= \frac{1}{12} \sum_{n=1}^{\infty} \left( \frac{15}{6n-5} - \frac{27}{4n-3} + \frac{48}{6n-4} - \frac{30}{8n-5} \right) \\
&= \frac{31\pi}{48\sqrt{3}} - \frac{5\pi}{16\sqrt{2}} - \frac{\pi}{8} + \frac{21}{16} \ln 3 - \frac{121}{48} \ln 2
\end{aligned}$$



$$+ \frac{5}{8\sqrt{2}} \ln(\sqrt{2} + 1).$$

于定理16(11), 令  $A=1$ ,  $B=6$ ,  $D=4$ ,  $a=1$ ,  $b=1$ ,  $d=2$ ,  $e=3$ ,  $k=2$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(4n-3)(6n-5)(6n-4)(8n-5)} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{3}{6n-5} - \frac{6}{4n-3} + \frac{12}{6n-4} - \frac{8}{8n-5} \right) \\ &= \frac{7\pi}{8\sqrt{3}} - \frac{\pi}{2\sqrt{2}} - \frac{\pi}{8} + \frac{15}{8} \ln 3 - \frac{15}{4} \ln 2 \\ & \quad + \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1); \end{aligned}$$

于定理 16 (11) 令  $A=3$ ,  $B=20$ ,  $D=15$ ,  $a=1$ ,  $b=1$ ,  $d=2$ ,  $e=3$ ,  $k=2$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(4n-3)(6n-5)(6n-4)(8n-5)} \\ &= \frac{1}{5} \sum_{n=1}^{\infty} \left( \frac{9}{6n-5} - \frac{20}{4n-3} + \frac{45}{6n-4} - \frac{32}{8n-5} \right) \\ &= \frac{2\pi\sqrt{3}}{5} - \frac{2\pi\sqrt{2}}{5} - \frac{\pi}{10} + \frac{27}{10} \ln 3 - \frac{28}{5} \ln 2 \\ & \quad + \frac{4\sqrt{2}}{5} \ln(\sqrt{2} + 1). \end{aligned}$$

根据例 3 中的结果, 有

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(4n-1)(6n-2)(6n-1)(8n-1)} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n^2}{(3n-1)(4n-1)(6n-1)(8n-1)} \\ &= -\frac{19\pi}{240\sqrt{3}} + \frac{\pi}{80\sqrt{2}} + \frac{3\pi}{80} + \frac{9}{80} \ln 3 - \frac{37}{240} \ln 2 \end{aligned}$$

$$- \frac{1}{40\sqrt{2}} \ln(\sqrt{2} + 1).$$

于是

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(2n-1)(3n-2)(3n-1)(4n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(4n-3)(6n-5)(6n-4)(8n-5)} \right. \\ & \quad \left. - \frac{4n^2}{(4n-1)(6n-2)(6n-1)(8n-1)} \right] \\ &= \frac{\pi\sqrt{3}}{5} - \frac{\pi}{10\sqrt{2}} - \frac{\pi}{4} - \frac{1}{15} \ln 2 \\ & \quad + \frac{1}{5\sqrt{2}} \ln(\sqrt{2} + 1), \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-2)(3n-1)(4n-1)} \\ &= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(4n-3)(6n-5)(6n-4)(8n-5)} \right. \\ & \quad \left. + \frac{4n^2}{(4n-1)(6n-2)(6n-1)(8n-1)} \right] \\ &= -\frac{\pi}{30\sqrt{3}} + \frac{\pi}{20} + \frac{9}{10} \ln 3 - \frac{13}{10} \ln 2. \end{aligned}$$

例6 求和:

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)}.$$

解 于定理16(3)令  $A=1, B=6, D=3, a=1, b=3, d=5, e=7, k=2$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(4n-1)(6n-1)(8n-1)} \\ &= \frac{1}{24} \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{6}{4n-1} + \frac{9}{6n-1} - \frac{4}{8n-1} \right) \end{aligned}$$

$$= -\frac{\pi\sqrt{3}}{32} + \frac{\pi}{48\sqrt{2}} + \frac{\pi}{24} + \frac{3}{32} \ln 3 - \frac{5}{48} \ln 2 \\ - \frac{1}{24\sqrt{2}} \ln(\sqrt{2} + 1).$$

由此以及例 3 中的相应结果即可求得

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(4n-1)(6n-1)(8n-1)(12n-1)} \\ = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(2n-1)(4n-1)(6n-1)(8n-1)} \right. \\ \left. - \frac{6n^2}{(4n-1)(6n-1)(8n-1)(12n-1)} \right] \\ = -\frac{11\pi\sqrt{3}}{320} + \frac{\pi}{24\sqrt{2}} + \frac{29\pi}{960} + \frac{33}{320} \ln 3 - \frac{13}{120} \ln 2 \\ + \frac{\sqrt{3}}{160} \ln(2 + \sqrt{3}) - \frac{1}{12\sqrt{2}} \ln(\sqrt{2} + 1).$$

例 7 求和:

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)}.$$

解 于定理 16(4) 令  $A=5, B=4, D=5, a=1, b=2, d=5, e=7, k=1$ , 即可推得

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(6n-1)(8n-1)} \\ = \frac{1}{60} \sum_{n=1}^{\infty} \left( \frac{5}{2n-1} - \frac{12}{3n-1} + \frac{15}{6n-1} - \frac{8}{8n-1} \right) \\ = -\frac{7\pi}{240\sqrt{3}} + \frac{\pi}{60\sqrt{2}} + \frac{\pi}{120} + \frac{1}{10} \ln 2 - \frac{3}{80} \ln 3 \\ - \frac{1}{30\sqrt{2}} \ln(\sqrt{2} + 1).$$

于定理 16(5) 令  $A=2, B=15, D=9, a=2, b=5, d=7$ ,

$e = 11, k = 1$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(3n-1)(6n-1)(8n-1)(12n-1)} \\ &= \frac{1}{90} \sum_{n=1}^{\infty} \left( \frac{2}{3n-1} - \frac{15}{6n-1} + \frac{18}{8n-1} - \frac{5}{12n-1} \right) \\ &= \frac{97\pi}{2160\sqrt{3}} - \frac{\pi}{40\sqrt{2}} - \frac{17\pi}{2160} + \frac{11}{360} \ln 2 - \frac{3}{80} \ln 3 \\ & \quad + \frac{1}{20\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{72\sqrt{3}} \ln(2+\sqrt{3}). \end{aligned}$$

于是

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(6n-1)(8n-1)(12n-1)} \\ &= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(2n-1)(3n-1)(6n-1)(8n-1)} \right. \\ & \quad \left. - \frac{6n^2}{(3n-1)(6n-1)(8n-1)(12n-1)} \right] \\ &= \frac{97\pi}{2160\sqrt{3}} - \frac{\pi}{40\sqrt{2}} - \frac{17\pi}{2160} + \frac{11}{360} \ln 2 - \frac{3}{80} \ln 3 \\ & \quad + \frac{1}{20\sqrt{2}} \ln(\sqrt{2}+1) - \frac{1}{72\sqrt{3}} \ln(2+\sqrt{3}). \end{aligned}$$

**例8** 求和:

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)}.$$

**解** 于定理 16(6) 令  $A=5, B=6, D=15, a=1,$

$b=2, d=3, e=7, k=1$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(8n-1)} \\ &= \frac{1}{30} \sum_{n=1}^{\infty} \left( \frac{5}{2n-1} - \frac{18}{3n-1} + \frac{15}{4n-1} - \frac{2}{8n-1} \right) \end{aligned}$$

$$= \frac{\pi}{10\sqrt{3}} + \frac{\pi}{120\sqrt{2}} - \frac{7\pi}{120} + \frac{61}{120} \ln 2 - \frac{3}{10} \ln 3 \\ - \frac{1}{60\sqrt{2}} \ln(\sqrt{2} + 1);$$

于定理16(7) 令  $A=8, B=15, D=6, a=2, b=3, d=7,$   
 $e=11, k=1$ , 即可推得

$$\sum_{n=1}^{\infty} \frac{n^2}{(3n-1)(4n-1)(8n-1)(12n-1)} \\ = \frac{1}{120} \sum_{n=1}^{\infty} \left( \frac{8}{3n-1} - \frac{15}{4n-1} + \frac{12}{8n-1} - \frac{5}{12n-1} \right) \\ = -\frac{17\pi}{2880\sqrt{3}} - \frac{\pi}{80\sqrt{2}} + \frac{37\pi}{2880} + \frac{9}{320} \ln 3 - \frac{13}{240} \ln 2 \\ + \frac{1}{40\sqrt{2}} \ln(\sqrt{2} + 1) - \frac{1}{96\sqrt{3}} \ln(2 + \sqrt{3}).$$

于是

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(8n-1)(12n-1)} \\ = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(2n-1)(3n-1)(4n-1)(8n-1)} \right. \\ \left. - \frac{6n^2}{(3n-1)(4n-1)(8n-1)(12n-1)} \right] \\ = \frac{13\pi}{480\sqrt{3}} + \frac{\pi}{60\sqrt{2}} - \frac{13\pi}{480} + \frac{1}{6} \ln 2 - \frac{3}{32} \ln 3 \\ + \frac{1}{80\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{30\sqrt{2}} \ln(\sqrt{2} + 1).$$

**例9** 求和:

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)}.$$

**解** 于定理16(8)令  $A=4, B=9, D=6, a=2, b=3,$   
 $d=5, e=11, k=1$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(3n-1)(4n-1)(6n-1)(12n-1)} \\
&= \frac{1}{36} \sum_{n=1}^{\infty} \left( \frac{4}{3n-1} - \frac{9}{4n-1} + \frac{6}{6n-1} - \frac{1}{12n-1} \right) \\
&= -\frac{49\pi}{864\sqrt{3}} + \frac{29\pi}{864} + \frac{3}{32} \ln 3 - \frac{5}{36} \ln 2 \\
&\quad - \frac{1}{144\sqrt{3}} \ln(2 + \sqrt{3}).
\end{aligned}$$

由此以及例 3 中的相应结果即可求得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)(12n-1)} \\
&= \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(2n-1)(3n-1)(4n-1)(6n-1)} \right. \\
&\quad \left. - \frac{6n^2}{(3n-1)(4n-1)(6n-1)(12n-1)} \right] \\
&= \frac{41\pi}{360\sqrt{3}} - \frac{47\pi}{720} + \frac{7}{20} \ln 2 - \frac{9}{40} \ln 3 \\
&\quad + \frac{1}{120\sqrt{3}} \ln(2 + \sqrt{3}).
\end{aligned}$$

仿此求得以下一系列的结果:

(一)于定理16(2)令  $A=33$ ,  $B=154$ ,  $D=21$ ,  $a=2$ ,  
 $b=3$ ,  $d=5$ ,  $e=3$ ,  $k=1$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(6n-1)(8n-5)} \\
&= -\frac{1}{231} \sum_{n=1}^{\infty} \left( \frac{99}{3n-1} - \frac{154}{4n-1} + \frac{63}{6n-1} - \frac{40}{8n-5} \right) \\
&= \frac{43\pi}{308\sqrt{3}} + \frac{5\pi}{231\sqrt{2}} - \frac{29\pi}{308} + \frac{229}{462} \ln 2
\end{aligned}$$

$$- \frac{87}{308} \ln 3 - \frac{5\sqrt{2}}{231} \ln(\sqrt{2} + 1);$$

于定理16(1), 令  $A=55$ ,  $B=36$ ,  $D=200$ ,  $a=3$ ,  $b=5$ ,  $d=3$ ,  $e=5$ ,  $k=2$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(4n-1)(6n-1)(8n-5)(12n-7)} \\ &= \frac{1}{1320} \sum_{n=1}^{\infty} \left( \frac{55}{4n-1} - \frac{36}{6n-1} + \frac{400}{8n-5} - \frac{693}{12n-7} \right) \\ &= \frac{17\pi\sqrt{3}}{704} + \frac{5\pi}{132\sqrt{2}} - \frac{239\pi}{3520} + \frac{7}{165} \ln 2 - \frac{51}{704} \ln 3 \\ &\quad - \frac{5}{66\sqrt{2}} \ln(\sqrt{2} + 1) + \frac{7\sqrt{3}}{160} \ln(2 + \sqrt{3}). \end{aligned}$$

由此以及上面已求出的相应的结果, 即可顺次求得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-5)} \\ &= - \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)} \right. \\ &\quad \left. - \frac{4n}{(3n-1)(4n-1)(6n-1)(8n-5)} \right] \\ &= - \frac{195\pi}{616\sqrt{3}} + \frac{10\pi\sqrt{2}}{231} + \frac{19\pi}{154} - \frac{235}{231} \ln 2 \\ &\quad + \frac{459}{616} \ln 3 - \frac{20\sqrt{2}}{231} \ln(\sqrt{2} + 1), \\ & \sum_{n=1}^{\infty} \frac{n}{(3n-1)(4n-1)(6n-1)(8n-5)(12n-7)} \\ &= - \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n}{(3n-1)(4n-1)(6n-1)(8n-5)} \right. \\ &\quad \left. - \frac{4n}{(4n-1)(6n-1)(8n-5)(12n-7)} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{185\pi}{3696\sqrt{3}} + \frac{10\pi}{231\sqrt{2}} - \frac{1093\pi}{18480} - \frac{251}{2310} \ln 2 - \frac{3}{1232} \ln 3 \\
&\quad - \frac{10\sqrt{2}}{231} \ln(\sqrt{2} + 1) + \frac{7}{40\sqrt{3}} \ln(2 + \sqrt{3}). \\
&\sum_{n=1}^{\infty} \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-5)(12n-7)} \\
&= - \sum_{n=1}^{\infty} \left[ \frac{n}{(2n-1)(3n-1)(4n-1)(6n-1)(8n-5)} \right. \\
&\quad \left. - \frac{6n}{(3n-1)(4n-1)(6n-1)(8n-5)(12n-7)} \right] \\
&= \frac{95\pi}{154\sqrt{3}} + \frac{20\pi\sqrt{2}}{231} - \frac{1473\pi}{3080} + \frac{422}{1155} \ln 2 - \frac{117}{154} \ln 3 \\
&\quad - \frac{40\sqrt{2}}{231} \ln(\sqrt{2} + 1) + \frac{7\sqrt{3}}{20} \ln(2 + \sqrt{3}).
\end{aligned}$$

(二)于定理16(10), 令  $A=147$ ,  $B=50$ ,  $D=32$ ,  $a=1$ ,  $b=1$ ,  $d=2$ ,  $e=3$ ,  $k=2$ , 即可推得

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)} \\
&= \frac{1}{120} \sum_{n=1}^{\infty} \left( \frac{147}{8n-7} - \frac{150}{6n-5} + \frac{96}{6n-4} - \frac{75}{8n-5} \right) \\
&= -\frac{59\pi}{240\sqrt{3}} + \frac{3\pi}{40\sqrt{2}} + \frac{37\pi}{320} - \frac{9}{80} \ln 3 - \frac{7}{60} \ln 2 \\
&\quad + \frac{37}{80\sqrt{2}} \ln(\sqrt{2} + 1);
\end{aligned}$$

于定理 16 (12), 令  $A=1210$ ,  $B=3087$ ,  $D=448$ ,  $a=1$ ,  $b=1$ ,  $d=2$ ,  $e=3$ ,  $k=2$ , 即可推得

$$\sum_{n=1}^{\infty} \frac{n^2}{(6n-4)(8n-7)(8n-5)(12n-11)}$$



$$\begin{aligned}
&= \frac{1}{5040} \sum_{n=1}^{\infty} \left( \frac{3630}{12n-11} - \frac{3087}{8n-7} + \frac{1344}{6n-4} - \frac{1125}{8n-5} \right) \\
&= \frac{2263\pi}{20160\sqrt{3}} - \frac{117\pi}{1120\sqrt{2}} + \frac{1439\pi}{40320} + \frac{351}{2240} \ln 3 \\
&\quad - \frac{799}{3360} \ln 2 + \frac{121}{672\sqrt{3}} \ln(2 + \sqrt{3}) \\
&\quad - \frac{109}{1120\sqrt{2}} \ln(\sqrt{2} + 1);
\end{aligned}$$

于定理16(9), 令  $A=121$ ,  $B=49$ ,  $D=25$ ,  $a=1$ ,  $b=1$ ,  $d=3$ ,  $e=5$ ,  $k=4$ , 即可推得

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^2}{(8n-7)(8n-5)(12n-11)(12n-7)} \\
&= \frac{1}{448} \sum_{n=1}^{\infty} \left( \frac{121}{12n-11} - \frac{98}{8n-7} + \frac{50}{8n-5} - \frac{49}{12n-7} \right) \\
&= \frac{85\pi}{1792\sqrt{3}} - \frac{3\pi}{224\sqrt{2}} - \frac{13\pi}{1792} + \frac{9}{448} \ln 3 - \frac{3}{224} \ln 2 \\
&\quad + \frac{85}{896\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{37}{448\sqrt{2}} \ln(\sqrt{2} + 1);
\end{aligned}$$

于定理16(10), 令  $A=27$ ,  $B=8$ ,  $D=2$ ,  $a=5$ ,  $b=4$ ,  $d=5$ ,  $e=7$ ,  $k=2$ , 即可推得

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)} \\
&= \frac{1}{120} \sum_{n=1}^{\infty} \left( \frac{27}{8n-3} - \frac{24}{6n-2} + \frac{6}{6n-1} - \frac{3}{8n-1} \right) \\
&= \frac{\pi}{240\sqrt{3}} - \frac{\pi}{40\sqrt{2}} + \frac{\pi}{64} + \frac{7}{60} \ln 2 - \frac{3}{80} \ln 3 \\
&\quad - \frac{1}{16\sqrt{2}} \ln(\sqrt{2} + 1);
\end{aligned}$$

于定理16(12), 令  $A=250$ ,  $B=567$ ,  $D=28$ ,  $a=7$ ,  $b=5$ ,

$d=5, e=7, k=2$ , 即可推得

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{n^2}{(6n-1)(8n-3)(8n-1)(12n-5)} \\
 &= \frac{1}{5040} \sum_{n=1}^{\infty} \left( \frac{750}{12n-5} - \frac{567}{8n-3} + \frac{84}{6n-1} - \frac{45}{8n-1} \right) \\
 &= \frac{97\pi}{6720\sqrt{3}} + \frac{17\pi}{1120\sqrt{2}} - \frac{761\pi}{40320} + \frac{51}{2240} \ln 3 \\
 &\quad - \frac{181}{10080} \ln 2 - \frac{25}{672\sqrt{3}} \ln(2+\sqrt{3}) \\
 &\quad + \frac{29}{1120\sqrt{2}} \ln(\sqrt{2}+1);
 \end{aligned}$$

于定理16(9), 令  $A=25, B=9, D=1, a=7, b=5, d=7, e=11, k=4$ , 即可推得

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{n^2}{(8n-3)(8n-1)(12n-5)(12n-1)} \\
 &= \frac{1}{448} \sum_{n=1}^{\infty} \left( \frac{25}{12n-5} - \frac{18}{8n-3} + \frac{2}{8n-1} - \frac{1}{12n-1} \right) \\
 &= \frac{13\pi}{1792\sqrt{3}} + \frac{\pi}{224\sqrt{2}} - \frac{13\pi}{1792} + \frac{3}{448} \ln 3 - \frac{1}{224} \ln 2 \\
 &\quad + \frac{5}{448\sqrt{2}} \ln(\sqrt{2}+1) - \frac{13}{896\sqrt{3}} \ln(2+\sqrt{3}).
 \end{aligned}$$

由此以及上面已求出的相应结果, 可顺次求得

$$\begin{aligned}
 & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)} \\
 &= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)} \right. \\
 &\quad \left. - \frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)} \right]
 \end{aligned}$$

$$= -\frac{\pi}{5\sqrt{3}} + \frac{\pi}{5\sqrt{2}} - \frac{1}{3} \ln 2 + \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)} + \frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)} \right]$$

$$= -\frac{\pi}{6\sqrt{3}} + \frac{\pi}{8} + \frac{3}{5} \ln 2 - \frac{3}{10} \ln 3;$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-4)(8n-7)(8n-5)(12n-11)} - \frac{4n^2}{(6n-1)(8n-3)(8n-1)(12n-5)} \right]$$

$$= \frac{\pi}{45\sqrt{3}} - \frac{17\pi}{140\sqrt{2}} + \frac{25\pi}{252} - \frac{1}{45} \ln 2 + \frac{25}{84\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{29}{140\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(3n-1)(4n-3)(4n-1)(6n-5)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-4)(8n-7)(8n-5)(12n-11)} + \frac{4n^2}{(6n-1)(8n-3)(8n-1)(12n-5)} \right]$$

$$= \frac{347\pi}{2520\sqrt{3}} - \frac{29\pi}{560} + \frac{51}{280} \ln 3 - \frac{209}{1260} \ln 2;$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(8n-7)(8n-5)(12n-11)(12n-7)} - \frac{4n^2}{(8n-3)(8n-1)(12n-5)(12n-1)} \right]$$

$$= -\frac{\pi}{28\sqrt{2}} + \frac{\pi}{28} + \frac{13}{112\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{5}{56\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(8n-7)(8n-5)(12n-11)(12n-7)} + \frac{4n^2}{(8n-3)(8n-1)(12n-5)(12n-1)} \right]$$

$$= \frac{13\pi}{224\sqrt{3}} - \frac{5\pi}{224} + \frac{3}{56} \ln 3 - \frac{1}{28} \ln 2.$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)}$$

$$= - \sum_{n=1}^{\infty} \left[ \frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)} - \frac{2n^2}{(6n-4)(8n-7)(8n-5)(12n-11)} \right]$$

$$= \frac{4741\pi}{10080\sqrt{3}} - \frac{159\pi}{560\sqrt{2}} - \frac{223\pi}{5040} + \frac{477}{1120} \ln 3$$

$$- \frac{201}{560} \ln 2 + \frac{121}{336\sqrt{3}} \ln(2 + \sqrt{3})$$

$$- \frac{23}{35\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{n^2}{(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. - \frac{2n^2}{(8n-7)(8n-5)(12n-11)(12n-7)} \right] \\
&= \frac{701\pi}{40320\sqrt{3}} - \frac{87\pi}{1120\sqrt{2}} + \frac{253\pi}{5040} + \frac{261}{2240} \ln 3 \\
&\quad - \frac{709}{3360} \ln 2 - \frac{13}{1344\sqrt{3}} \ln(2 + \sqrt{3}) \\
&\quad + \frac{19}{280\sqrt{2}} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \\
&= - \sum_{n=1}^{\infty} \left[ \frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)} \right. \\
&\quad \left. - \frac{2n^2}{(6n-1)(8n-3)(8n-1)(12n-5)} \right] \\
&= \frac{83\pi}{3360\sqrt{3}} + \frac{31\pi}{560\sqrt{2}} - \frac{269\pi}{5040} + \frac{93}{1120} \ln 3 \\
&\quad - \frac{769}{5040} \ln 2 - \frac{25}{336\sqrt{3}} \ln(2 + \sqrt{3}) \\
&\quad + \frac{2\sqrt{2}}{35} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{n^2}{(6n-1)(8n-3)(8n-1)(12n-5)} \right. \\
&\quad \left. - \frac{2n^2}{(8n-3)(8n-1)(12n-5)(12n-1)} \right]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\pi}{13440\sqrt{3}} + \frac{\pi}{160\sqrt{2}} - \frac{11\pi}{2520} + \frac{3}{320} \ln 3 \\
&\quad - \frac{13}{1440} \ln 2 - \frac{11}{1344\sqrt{3}} \ln(2 + \sqrt{3}) \\
&\quad + \frac{1}{280\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. - \frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \right] \\
&= \frac{11\pi}{45\sqrt{3}} - \frac{31\pi}{70\sqrt{2}} + \frac{25\pi}{126} + \frac{13}{45} \ln 2 \\
&\quad + \frac{25}{42\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{16\sqrt{2}}{35} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right. \\
&\quad \left. + \frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \right] \\
&= \frac{557\pi}{1260\sqrt{3}} - \frac{8\pi}{35} + \frac{93}{140} \ln 3 - \frac{587}{630} \ln 2,
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} \\
&= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\
&\quad \left. - \frac{4n^2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right]
\end{aligned}$$

$$= \frac{\pi}{45\sqrt{3}} - \frac{\pi}{20\sqrt{2}} + \frac{\pi}{36} - \frac{1}{45} \ln 2$$

$$+ \frac{11}{168\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{1}{35\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)}$$

$$= \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right.$$

$$\left. + \frac{4n^2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right]$$

$$= \frac{109\pi}{5040\sqrt{3}} - \frac{\pi}{140} + \frac{3}{40} \ln 3 - \frac{119}{1260} \ln 2.$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)}$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)} \right.$$

$$\left. - \frac{2n^2}{(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right]$$

$$= \frac{2927\pi}{20160\sqrt{3}} - \frac{3\pi}{70\sqrt{2}} - \frac{27\pi}{560} + \frac{9}{140} \ln 3$$

$$+ \frac{53}{2520} \ln 2 + \frac{85}{672\sqrt{3}} \ln(2 + \sqrt{3})$$

$$- \frac{37}{140\sqrt{2}} \ln(\sqrt{2} + 1),$$

$$\sum_{n=1}^{\infty} \frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)}$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \left[ \frac{n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)} \right.$$

$$\begin{aligned}
& - \frac{2n^2}{(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \Big] \\
& = \frac{167\pi}{20160\sqrt{3}} + \frac{\pi}{70\sqrt{2}} - \frac{5\pi}{336} + \frac{3}{140} \ln 3 \\
& \quad - \frac{113}{2520} \ln 2 - \frac{13}{672\sqrt{3}} \ln(2 + \sqrt{3}) \\
& \quad + \frac{1}{28\sqrt{2}} \ln(\sqrt{2} + 1).
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\
& \quad \left. - \frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right]
\end{aligned}$$

$$\begin{aligned}
& = \frac{\pi}{15\sqrt{3}} - \frac{2\pi\sqrt{2}}{35} + \frac{\pi}{21} + \frac{1}{9} \ln 2 \\
& \quad + \frac{13}{84\sqrt{3}} \ln(2 + \sqrt{3}) - \frac{\sqrt{2}}{7} \ln(\sqrt{2} + 1),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n^2}{(3n-2)(3n-1)(4n-3)(4n-1)(6n-5)(6n-1)} \\
& = \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^2}{(6n-5)(6n-4)(8n-7)(8n-5)(12n-11)(12n-7)} \right. \\
& \quad \left. + \frac{4n^2}{(6n-2)(6n-1)(8n-3)(8n-1)(12n-5)(12n-1)} \right]
\end{aligned}$$

$$= \frac{67\pi}{504\sqrt{3}} - \frac{\pi}{14} + \frac{6}{35} \ln 3 - \frac{26}{105} \ln 2.$$



## 四、一般的结果

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ \left. + \frac{1}{k(mn-m+2)+a} + \cdots + \frac{1}{k(mn-2)+a} \right. \\ \left. - \frac{m-1}{k(mn-1)+a} \right]$$

**定理17** 设  $m$  为正整数, 且  $m \geq 2$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ \left. + \frac{1}{k(mn-m+2)+a} + \cdots + \frac{1}{k(mn-2)+a} \right. \\ \left. - \frac{m-1}{k(mn-1)+a} \right] \\ = C(a, k) - mC(a+mk-k, mk) \\ + \frac{1}{k} \ln \frac{a+mk-k}{a}.$$

**证明** 根据定理 2, 有

$$\sum_{i=1}^n \left[ \frac{1}{k(m_i-m)+a} + \frac{1}{k(m_i-m+1)+a} \right]$$

$$\begin{aligned}
& + \frac{1}{k(mi-m+2)+a} + \cdots + \frac{1}{k(mi-2)+a} \\
& - \frac{m-1}{k(mi-1)+a} \Big] \\
& = C(a, mk) + C(a+k, mk) + C(a+2k, mk) + \cdots \\
& + C(a+mk-2k, mk) - (m-1)C(a+mk-k, mk) \\
& + \frac{1}{mk} \ln \frac{(mnk-mk+a) \cdots (mnk-2k+a)}{(mnk-k+a)^{m-1}} \\
& - \frac{1}{mk} \ln \frac{a(a+k)(a+2k) \cdots (a+mk-2k)}{(a+mk-k)^{m-1}} \\
& + \varepsilon_n(a, mk) + \varepsilon_n(a+k, mk) + \varepsilon_n(a+2k, mk) \\
& + \cdots + \varepsilon_n(a+mk-2k, mk) \\
& - (m-1)\varepsilon_n(a+mk-k, mk).
\end{aligned}$$

令  $n \rightarrow \infty$  并取极限即得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\
& + \frac{1}{k(mn-m+2)+a} + \cdots + \frac{1}{k(mn-2)+a} \\
& \left. - \frac{m-1}{k(mn-1)+a} \right] \\
& = C(a, mk) + C(a+k, mk) + C(a+2k, mk) + \cdots \\
& + C(a+mk-2k, mk) - (m-1)C(a+mk-k, mk) \\
& - \frac{1}{mk} \ln \frac{a(a+k)(a+2k) \cdots (a+mk-2k)}{(a+mk-k)^{m-1}}.
\end{aligned}$$

根据定理 4, 有

$$\begin{aligned}
& C(a, mk) + C(a+k, mk) + C(a+2k, mk) \\
& + \cdots + C(a+mk-2k, mk)
\end{aligned}$$

$$= C(a, k) - C(a + mk - k, mk) \\ + \frac{1}{mk} \ln \frac{(a+k)(a+2k)\cdots(a+mk-k)}{a^{m-1}}.$$

$$\text{于是 } \sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ \left. + \frac{1}{k(mn-m+2)+a} + \cdots + \frac{1}{k(mn-2)+a} \right. \\ \left. - \frac{m-1}{k(mn-1)+a} \right]$$

$$= C(a, k) - mC(a + mk - k, mk) \\ + \frac{1}{mk} \ln \frac{(a+k)(a+2k)\cdots(a+mk-k)}{a^{m-1}} \\ - \frac{1}{mk} \ln \frac{a(a+k)(a+2k)\cdots(a+mk-2k)}{(a+mk-k)^{m-1}}$$

$$= C(a, k) - mC(a + mk - k, mk) + \frac{1}{k} \ln \frac{a + mk - k}{a}.$$

**定理17'** 设  $m$  为正整数, 且  $m \geq 2$ , 则

$$\sum_{n=1}^{\infty} \left( \frac{1}{mn-m+1} + \frac{1}{mn-m+2} + \frac{1}{mn-m+3} + \cdots \right. \\ \left. + \frac{1}{mn-1} - \frac{m-1}{mn} \right) = \ln m.$$

**证明** 根据命题 1,  $mC(m, m) = C(1, 1)$ . 于定理 17 令  $a=1$ ,  $k=1$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{mn-m+1} + \frac{1}{mn-m+2} + \frac{1}{mn-m+3} \right. \\ \left. + \cdots + \frac{1}{mn-1} - \frac{m-1}{mn} \right) \\ = C(1, 1) - mC(m, m) + \ln m = \ln m.$$

于定理17', 令  $m=2, 3, 4, \dots$  则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{2n} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \ln 2,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{2}{3n} \right) = \ln 3,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{4n-3} + \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{3}{4n} \right) = \ln 4,$$

...

于定理17 令  $m=3, a=3, k=1$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{3n} + \frac{1}{3n+1} - \frac{2}{3n+2} \right) = \frac{\pi}{2\sqrt{3}} - \frac{1}{2} \ln 3,$$

令  $m=3, a=1, k=2$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{6n-5} + \frac{1}{6n-3} - \frac{2}{6n-1} \right) = \frac{\pi\sqrt{3}}{4} - \frac{1}{4} \ln 3,$$

令  $m=3, a=1, k=4$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} + \frac{1}{12n-7} - \frac{2}{12n-3} \right) = \frac{\pi}{4} + \frac{1}{4} \ln 3,$$

令  $m=3, a=5, k=8$ , 即可推得

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{1}{24n-19} + \frac{1}{24n-11} - \frac{2}{24n-3} \right) \\ = \frac{\pi}{8} + \frac{1}{8} \ln 3 - \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

令  $m=4, a=3, k=1$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{4n-1} + \frac{1}{4n} + \frac{1}{4n+1} - \frac{3}{4n+2} \right) = \frac{1}{2},$$

令  $m=4, a=1, k=2$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{8n-7} + \frac{1}{8n-5} + \frac{1}{8n-3} - \frac{3}{8n-1} \right) \\ = \frac{\pi}{2\sqrt{2}} + \frac{\pi}{4} - \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1);$$

令  $m=4$ ,  $a=5$ ,  $k=2$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{8n-3} + \frac{1}{8n-1} + \frac{1}{8n+1} - \frac{3}{8n+3} \right) \\ = -\frac{\pi}{2\sqrt{2}} + \frac{\pi}{4} + \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1);$$

令  $m=4$ ,  $a=1$ ,  $k=3$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} + \frac{1}{12n-8} + \frac{1}{12n-5} - \frac{3}{12n-2} \right) = \frac{2\pi}{3\sqrt{3}},$$

令  $m=4$ ,  $a=2$ ,  $k=3$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-10} + \frac{1}{12n-7} + \frac{1}{12n-4} - \frac{3}{12n-1} \right) \\ = \frac{\pi}{3\sqrt{3}} + \frac{\pi}{3} - \frac{1}{3} \ln 2 - \frac{1}{\sqrt{3}} \ln(2 + \sqrt{3}).$$

仿照定理17的证法, 即可证明以下一系列的定理.

**定理18** 设  $m$  为正整数, 且  $m \geq 3$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ \left. + \frac{1}{k(mn-m+2)+a} + \cdots + \frac{1}{k(mn-3)+a} \right. \\ \left. - \frac{m-1}{k(mn-2)+a} + \frac{1}{k(mn-1)+a} \right] \\ = C(a, k) - mC(a+mk-2k, mk) \\ + \frac{1}{k} \ln \frac{a+mk-2k}{a}.$$

于定理18令  $a = k - 1$ ,  $m = k + 1$ , 则有

**定理18'** 设  $k$  为正整数, 且  $k \geq 2$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k)-1} + \frac{1}{k(kn+n-k+1)-1} \right. \\ & \quad + \frac{1}{k(kn+n-k+2)-1} + \cdots + \frac{1}{k(kn+n-2)-1} \\ & \quad \left. - \frac{k}{k(kn+n-1)-1} + \frac{1}{k(kn+n)-1} \right] \\ & = \frac{1}{k} \ln(k+1). \end{aligned}$$

于定理18' 令  $k = 2$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{6n-5} - \frac{2}{6n-3} + \frac{1}{6n-1} \right) = \frac{1}{2} \ln 3,$$

于是 
$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-3)(6n-1)} \\ & = \frac{1}{8} \sum_{n=1}^{\infty} \left( \frac{1}{6n-5} - \frac{2}{6n-3} + \frac{1}{6n-1} \right) = \frac{1}{16} \ln 3, \end{aligned}$$

令  $k = 3, 4$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{12n-10} + \frac{1}{12n-7} - \frac{3}{12n-4} + \frac{1}{12n-1} \right) \\ & = \frac{2}{3} \ln 2, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{20n-17} + \frac{1}{20n-13} + \frac{1}{20n-9} - \frac{4}{20n-5} + \frac{1}{20n-1} \right) \\ & = \frac{1}{4} \ln 5. \end{aligned}$$

于定理18令  $m = 3$ ,  $a = 1$ ,  $k = 1$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{3n-2} - \frac{2}{3n-1} + \frac{1}{3n} \right) = \frac{\pi}{2\sqrt{3}} - \frac{1}{2} \ln 3,$$

$$\begin{aligned}
 \text{于是 } & \sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)3n} \\
 &= \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{3n-2} - \frac{2}{3n-1} + \frac{1}{3n} \right) \\
 &= \frac{\pi}{4\sqrt{3}} - \frac{1}{4} \ln 3,
 \end{aligned}$$

令  $m=3$ ,  $a=1$ ,  $k=4$ , 即可推得

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{2}{12n-7} + \frac{1}{12n-3} \right) \\
 &= \frac{\pi\sqrt{3}}{8} - \frac{\pi}{8} - \frac{1}{8} \ln 3 + \frac{\sqrt{3}}{4} \ln(2+\sqrt{3}),
 \end{aligned}$$

$$\begin{aligned}
 \text{于是 } & \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-7)(12n-3)} \\
 &= \frac{1}{32} \sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{2}{12n-7} + \frac{1}{12n-3} \right) \\
 &= \frac{\pi\sqrt{3}}{256} - \frac{\pi}{256} - \frac{1}{256} \ln 3 + \frac{\sqrt{3}}{128} \ln(2+\sqrt{3}),
 \end{aligned}$$

令  $m=3$ ,  $a=3$ ,  $k=4$ , 即可推得

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \left( \frac{1}{12n-9} - \frac{2}{12n-5} + \frac{1}{12n-1} \right) \\
 &= -\frac{\pi\sqrt{3}}{8} + \frac{\pi}{8} - \frac{1}{8} \ln 3 + \frac{\sqrt{3}}{4} \ln(2+\sqrt{3}),
 \end{aligned}$$

$$\begin{aligned}
 \text{于是 } & \sum_{n=1}^{\infty} \frac{1}{(12n-9)(12n-5)(12n-1)} \\
 &= \frac{1}{32} \sum_{n=1}^{\infty} \left( \frac{1}{12n-9} - \frac{2}{12n-5} + \frac{1}{12n-1} \right) \\
 &= -\frac{\pi\sqrt{3}}{256} + \frac{\pi}{256} - \frac{1}{256} \ln 3 + \frac{\sqrt{3}}{128} \ln(2+\sqrt{3}),
 \end{aligned}$$

令  $m=3$ ,  $a=1$ ,  $k=8$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{2}{24n-15} + \frac{1}{24n-7} \right) \\ = \frac{\pi}{8} + \frac{1}{8} \ln 3 + \frac{1}{2\sqrt{2}} \ln (\sqrt{2} + 1),$$

于是 
$$\sum_{n=1}^{\infty} \frac{1}{(24n-23)(24n-15)(24n-7)} \\ = \frac{1}{128} \sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{2}{24n-15} + \frac{1}{24n-7} \right) \\ = \frac{\pi}{1024} + \frac{1}{1024} \ln 3 + \frac{1}{256\sqrt{2}} \ln (\sqrt{2} + 1),$$

令  $m=4$ ,  $a=1$ ,  $k=1$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{4n-3} + \frac{1}{4n-2} - \frac{3}{4n-1} + \frac{1}{4n} \right) \\ = \frac{\pi}{2} - \ln 2,$$

令  $m=4$ ,  $a=1$ ,  $k=2$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{8n-7} + \frac{1}{8n-5} - \frac{3}{8n-3} + \frac{1}{8n-1} \right) \\ = \frac{\pi}{2\sqrt{2}} - \frac{\pi}{4} + \frac{1}{\sqrt{2}} \ln (\sqrt{2} + 1),$$

令  $m=4$ ,  $a=1$ ,  $k=3$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} + \frac{1}{12n-8} - \frac{3}{12n-5} + \frac{1}{12n-2} \right) \\ = -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{3} - \frac{1}{3} \ln 2 + \frac{1}{\sqrt{3}} \ln (2 + \sqrt{3}).$$

**定理19** 设  $m$  为正整数, 且  $m \geq 4$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right]$$



$$\begin{aligned}
& + \frac{1}{k(mn-m+2)+a} + \cdots + \frac{1}{k(mn-4)+a} \\
& - \frac{m-1}{k(mn-3)+a} + \frac{1}{k(mn-2)+a} + \frac{1}{k(mn-1)+a} \Big] \\
& = C(a, k) - mC(a+mk-3k, mk) \\
& + \frac{1}{k} \ln \frac{a+mk-3k}{a}.
\end{aligned}$$

于定理19令  $a = k-1$ ,  $m = 2k+1$ , 则有

**定理19'** 设  $k$  为正整数, 且  $k \geq 2$ , 则

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{1}{k(2kn+n-2k)-1} + \frac{1}{k(2kn+n-2k+1)-1} \right. \\
& + \frac{1}{k(2kn+n-2k+2)-1} + \cdots + \frac{1}{k(2kn+n-3)-1} \\
& - \frac{2k}{k(2kn+n-2)-1} + \frac{1}{k(2kn+n-1)-1} \\
& \left. + \frac{1}{k(2kn+n)-1} \right] \\
& = \frac{1}{k} \ln(2k+1).
\end{aligned}$$

于定理19'令  $k = 2, 3$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left( \frac{1}{10n-9} + \frac{1}{10n-7} - \frac{4}{10n-5} + \frac{1}{10n-3} + \frac{1}{10n-1} \right) \\
& = \frac{1}{2} \ln 5; \\
& \sum_{n=1}^{\infty} \left( \frac{1}{21n-19} + \frac{1}{21n-16} + \frac{1}{21n-13} + \frac{1}{21n-10} - \frac{6}{21n-7} \right. \\
& \quad \left. + \frac{1}{21n-4} + \frac{1}{21n-1} \right) \\
& = \frac{1}{3} \ln 7.
\end{aligned}$$

于定理19令  $a = k - 2$ ,  $m = k + 1$ , 则有

定理19'' 设  $k$  为正整数, 且  $k \geq 3$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k)-2} + \frac{1}{k(kn+n-k+1)-2} \right. \\ & \quad + \frac{1}{k(kn+n-k+2)-2} + \cdots + \frac{1}{k(kn+n-3)-2} \\ & \quad - \frac{k}{k(kn+n-2)-2} + \frac{1}{k(kn+n-1)-2} \\ & \quad \left. + \frac{1}{k(kn+n)-2} \right] \\ & = \frac{1}{k} \ln(k+1). \end{aligned}$$

于定理19''令  $k = 3, 4, 5$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{3}{12n-8} + \frac{1}{12n-5} + \frac{1}{12n-2} \right) \\ & = \frac{2}{3} \ln 2, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{10n-9} + \frac{1}{10n-7} - \frac{2}{12n-5} + \frac{1}{10n-3} + \frac{1}{10n-1} \right) \\ & = \frac{1}{2} \ln 5, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{30n-27} + \frac{1}{30n-22} + \frac{1}{30n-17} - \frac{5}{30n-12} \right. \\ & \quad \left. + \frac{1}{30n-7} + \frac{1}{30n-2} \right) \\ & = \frac{1}{5} \ln 6. \end{aligned}$$

于定理19令  $m = 4$ ,  $a = 2$ ,  $k = 3$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-10} - \frac{3}{12n-7} + \frac{1}{12n-4} + \frac{1}{12n-1} \right) \\ = \frac{\pi}{3\sqrt{3}} - \frac{\pi}{3} - \frac{1}{3} \ln 2 + \frac{1}{\sqrt{3}} \ln(2 + \sqrt{3}).$$

**定理20** 设  $m$  为正整数, 且  $m \geq 5$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ \left. + \frac{1}{k(mn-m+2)+a} + \cdots + \frac{1}{k(mn-5)+a} \right. \\ \left. - \frac{m-1}{k(mn-4)+a} + \frac{1}{k(mn-3)+a} + \frac{1}{k(mn-2)+a} \right. \\ \left. + \frac{1}{k(mn-1)+a} \right] \\ = C(a, k) - mC(a + mk - 4k, mk) \\ + \frac{1}{k} \ln \frac{a + mk - 4k}{a}.$$

于定理20令  $a = k - 1$ ,  $m = 3k + 1$ , 则有

**定理20'** 设  $k$  为正整数, 且  $k \geq 2$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(3kn+n-3k)-1} + \frac{1}{k(3kn+n-3k+1)-1} \right. \\ \left. + \frac{1}{k(3kn+n-3k+2)-1} + \cdots + \frac{1}{k(3kn+n-4)-1} \right. \\ \left. - \frac{3k}{k(3kn+n-3)-1} + \frac{1}{k(3kn+n-2)-1} \right. \\ \left. + \frac{1}{k(3kn+n-1)-1} + \frac{1}{k(3kn+n)-1} \right] \\ = \frac{1}{k} \ln(3k+1).$$

于定理20'令  $k = 2, 3$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{14n-13} + \frac{1}{14n-11} + \frac{1}{14n-9} - \frac{6}{14n-7} + \frac{1}{14n-5} \right. \\ \left. + \frac{1}{14n-3} + \frac{1}{14n-1} \right)$$

$$= \frac{1}{2} \ln 7;$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{30n-28} + \frac{1}{30n-25} + \cdots + \frac{1}{30n-13} - \frac{1}{30n-10} \right. \\ \left. + \frac{1}{30n-7} + \frac{1}{30n-4} + \frac{1}{30n-1} \right)$$

$$= \frac{1}{3} \ln 10.$$

于定理20令  $a = k - 3$ ,  $m = k + 1$ , 则有

**定理20''** 设  $k$  为正整数, 且  $k \geq 4$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k)-3} + \frac{1}{k(kn+n-k+1)-3} \right. \\ \left. + \frac{1}{k(kn+n-k+2)-3} + \cdots + \frac{1}{k(kn+n-4)-3} \right. \\ \left. - \frac{k}{k(kn+n-3)-3} + \frac{1}{k(kn+n-2)-3} \right. \\ \left. + \frac{1}{k(kn+n-1)-3} + \frac{1}{k(kn+n)-3} \right]$$

$$= \frac{1}{k} \ln(k+1).$$

于定理20''令  $k = 4, 5, 6$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{20n-19} - \frac{4}{20n-15} + \frac{1}{20n-11} + \frac{1}{20n-7} + \frac{1}{20n-3} \right)$$

$$= \frac{1}{4} \ln 5;$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{30n-28} + \frac{1}{30n-23} - \frac{5}{30n-18} + \frac{1}{30n-13} + \frac{1}{30n-8} + \frac{1}{30n-3} \right)$$

$$= \frac{1}{5} \ln 6;$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{14n-13} + \frac{1}{14n-11} + \frac{1}{14n-9} - \frac{6}{14n-7} + \frac{1}{14n-5} + \frac{1}{14n-3} + \frac{1}{14n-1} \right)$$

$$= \frac{1}{2} \ln 7.$$

于定理20令  $m=5$ ,  $a=2$ ,  $k=3$ , 即可推得

$$\sum_{n=1}^{\infty} \left( -\frac{1}{15n-13} + \frac{4}{15n-10} - \frac{1}{15n-7} - \frac{1}{15n-4} - \frac{1}{15n-1} \right) = \frac{\pi}{3\sqrt{3}} - \frac{1}{3} \ln 5.$$

**定理21** 设  $m$  为正整数, 且  $m \geq 6$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ & \quad + \frac{1}{k(mn-m+2)+a} + \cdots + \frac{1}{k(mn-6)+a} \\ & \quad - \frac{m-1}{k(mn-5)+a} + \frac{1}{k(mn-4)+a} + \cdots \\ & \quad \left. + \frac{1}{k(mn-1)+a} \right] \\ & = C(a, k) - mC(a+mk-5k, mk) \\ & \quad + \frac{1}{k} \ln \frac{a+mk-5k}{a}. \end{aligned}$$

于定理21令  $a = k - 1$ ,  $m = 4k + 1$ , 则有

**定理21'** 设  $k$  为正整数, 且  $k \geq 2$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(4kn + n - 4k) - 1} + \frac{1}{k(4kn + n - 4k + 1) - 1} \right. \\ & \quad + \frac{1}{k(4kn + n - 4k + 2) - 1} + \cdots + \frac{1}{k(4kn + n - 5) - 1} \\ & \quad - \frac{4k}{k(4kn + n - 4) - 1} + \frac{1}{k(4kn + n - 3) - 1} \\ & \quad \left. + \cdots + \frac{1}{k(4kn + n) - 1} \right] \\ & = \frac{1}{k} \ln(4k + 1). \end{aligned}$$

于定理21'令  $k = 2$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{18n - 17} + \frac{1}{18n - 15} + \frac{1}{18n - 13} + \frac{1}{18n - 11} - \frac{8}{18n - 9} \right. \\ & \quad \left. + \frac{1}{18n - 7} + \cdots + \frac{1}{18n - 1} \right) \\ & = \ln 3. \end{aligned}$$

于定理21令  $a = k - 2$ ,  $m = 2k + 1$ , 则有

**定理21''** 设  $k$  为正整数, 且  $k \geq 3$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(2kn + n - 2k) - 2} + \frac{1}{k(2kn + n - 2k + 1) - 2} \right. \\ & \quad + \frac{1}{k(2kn + n - 2k + 2) - 2} + \cdots + \frac{1}{k(2kn + n - 5) - 2} \\ & \quad - \frac{2k}{k(2kn + n - 4) - 2} + \frac{1}{k(2kn + n - 3) - 2} \\ & \quad \left. + \cdots + \frac{1}{k(2kn + n) - 2} \right] \end{aligned}$$

$$= \frac{1}{k} \ln(2k+1).$$

于定理21"令  $k=3, 4$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{21n-20} + \frac{1}{21n-17} - \frac{6}{21n-14} + \frac{1}{21n-11} \right. \\ \left. + \cdots + \frac{1}{21n-2} \right)$$

$$= \frac{1}{3} \ln 7,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{18n-17} + \frac{1}{18n-15} + \frac{1}{18n-13} + \frac{1}{18n-11} - \frac{8}{18n-9} \right. \\ \left. + \frac{1}{18n-7} + \cdots + \frac{1}{18n-1} \right)$$

$$= \ln 3.$$

于定理21令  $a=k-4$ ,  $k=k+1$ , 则有

**定理21'''** 设  $k$  为正整数, 且  $k \geq 5$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k)-4} - \frac{1}{k(kn+n-k+1)-4} \right. \\ \left. + \frac{1}{k(kn+n-k+2)-4} + \cdots + \frac{1}{k(kn+n-5)-4} \right. \\ \left. - \frac{k}{k(kn+n-4)-4} + \frac{1}{k(kn+n-3)-4} \right. \\ \left. + \cdots + \frac{1}{k(kn+n)-4} \right]$$

$$= \frac{1}{k} \ln(k+1).$$

于定理21'''令  $k=5, 6$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{30n-29} - \frac{5}{30n-24} + \frac{1}{30n-19} + \cdots + \frac{1}{30n-4} \right)$$

$$= \frac{1}{5} \ln 6,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{21n-20} + \frac{1}{21n-17} - \frac{6}{21n-14} + \frac{1}{21n-11} + \dots \right. \\ \left. + \frac{1}{21n-2} \right)$$

$$= \frac{1}{3} \ln 7.$$

**定理22** 设  $m$  为正整数, 且  $m \geq 7$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ \left. + \frac{1}{k(mn-m+2)+a} + \dots + \frac{1}{k(mn-7)+a} \right. \\ \left. - \frac{m-1}{k(mn-6)+a} + \frac{1}{k(mn-5)+a} + \dots \right. \\ \left. + \frac{1}{k(mn-1)+a} \right] \\ = C(a, k) - mC(a+mk-6k, mk) \\ + \frac{1}{k} \ln \frac{a+mk-6k}{a}.$$

于定理22令  $a = k-1$ ,  $m = 5k+1$ , 则有

**定理22'** 设  $k$  为正整数, 且  $k \geq 2$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(5kn+n-5k)-1} + \frac{1}{k(5kn+n-5k+1)-1} \right. \\ \left. + \frac{1}{k(5kn+n-5k+2)-1} + \dots + \frac{1}{k(5kn+n-6)-1} \right. \\ \left. - \frac{5k}{k(5kn+n-5)-1} + \frac{1}{k(5kn+n-4)-1} + \dots \right]$$



$$+ \frac{1}{k(5kn+n)-1} \Big] \\ = \frac{1}{k} \ln(5k+1).$$

于定理22' 令  $k=2$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{22n-21} + \frac{1}{22n-19} + \cdots + \frac{1}{22n-13} - \frac{10}{22n-11} \right. \\ \left. + \frac{1}{22n-9} + \cdots + \frac{1}{22n-1} \right) \\ = \frac{1}{2} \ln 11.$$

于定理22 令  $a=k-5$ ,  $m=k+1$ , 则有

**定理22''** 设  $k$  为正整数, 且  $k \geq 6$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k)-5} + \frac{1}{k(kn+n-k+1)-5} \right. \\ \left. + \frac{1}{k(kn+n-k+2)-5} + \cdots + \frac{1}{k(kn+n-6)-5} \right. \\ \left. - \frac{k}{k(kn+n-5)-5} + \frac{1}{k(kn+n-4)-5} \right. \\ \left. + \cdots + \frac{1}{k(kn+n)-5} \right] \\ = \frac{1}{k} \ln(k+1).$$

于定理22'' 令  $k=6, 7$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{42n-41} - \frac{6}{42n-35} + \frac{1}{42n-29} + \cdots + \frac{1}{42n-5} \right) \\ = \frac{1}{6} \ln 7;$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{56n-54} + \frac{1}{56n-47} - \frac{7}{56n+40} + \frac{1}{56n+33} \right. \\ \left. + \cdots + \frac{1}{56n-5} \right) \\ = \frac{3}{7} \ln 2.$$

**定理23** 设  $m$  为正整数, 且  $m \geq 8$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ \left. + \frac{1}{k(mn-m+2)+a} + \cdots + \frac{1}{k(mn-8)+a} \right. \\ \left. - \frac{m-1}{k(mn-7)+a} + \frac{1}{k(mn-6)+a} + \cdots \right. \\ \left. + \frac{1}{k(mn-1)+a} \right] \\ = C(a, k) - mC(a + mk - 7k, mk) \\ + \frac{1}{k} \ln \frac{a + mk - 7k}{a}.$$

于定理23令  $a = k - 1$ ,  $m = 6k + 1$ . 则有

**定理23'** 设  $k$  为正整数, 且  $k \geq 2$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(6kn+n-6k)-1} + \frac{1}{k(6kn+n-6k+1)-1} \right. \\ \left. + \frac{1}{k(6kn+n-6k+2)-1} + \cdots + \frac{1}{k(6kn+n-7)-1} \right. \\ \left. - \frac{6k}{k(6kn+n-6)-1} + \frac{1}{k(6kn+n-5)-1} + \cdots \right. \\ \left. + \frac{1}{k(6kn+n)-1} \right]$$

$$= \frac{1}{k} \ln(6k+1).$$

于定理23' 令  $k=2$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{26n-25} + \frac{1}{26n-23} + \cdots + \frac{1}{26n-15} - \frac{12}{26n-13} \right. \\ & \quad \left. + \frac{1}{26n-11} + \cdots + \frac{1}{26n-1} \right) \\ &= \frac{1}{2} \ln 13. \end{aligned}$$

于定理23 令  $a=k-2$ ,  $m=3k+1$ , 则有

**定理23''** 设  $k$  为正整数, 且  $k \geq 3$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(3kn+n-3k)-2} + \frac{1}{k(3kn+n-3k+1)-2} \right. \\ & \quad + \frac{1}{k(3kn+n-3k+2)-2} + \cdots + \frac{1}{k(3kn+n-7)-2} \\ & \quad - \frac{3k}{k(3kn+n-6)-2} + \frac{1}{k(3kn+n-5)-2} + \cdots \\ & \quad \left. + \frac{1}{k(3kn+n)-2} \right] \\ &= \frac{1}{k} \ln(3k+1). \end{aligned}$$

于定理23'' 令  $k=3, 4$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{30n-29} + \frac{1}{30n-26} + \frac{1}{30n-23} - \frac{9}{30n-20} \right. \\ & \quad \left. + \frac{1}{30n-17} + \cdots + \frac{1}{30n-2} \right) \\ &= \frac{1}{3} \ln 10, \end{aligned}$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{26n-25} + \frac{1}{26n-23} + \cdots + \frac{1}{26n-15} - \frac{12}{26n-13} \right.$$

$$+ \frac{1}{26n-11} + \cdots + \frac{1}{26n-1})$$

$$= \frac{1}{2} \ln 13.$$

于定理23令  $a = k - 3$ ,  $m = 2k + 1$ , 则有

**定理23'''** 设  $k$  为正整数, 且  $k \geq 4$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(2kn+n-2k)-3} + \frac{1}{k(2kn+n-2k+1)-3} \right. \\ + \frac{1}{k(2kn+n-2k+2)-3} + \cdots + \frac{1}{k(2kn+n-7)-3} \\ - \frac{2k}{k(2kn+n-6)-3} + \frac{1}{k(2kn+n-5)-3} \\ \left. + \cdots + \frac{1}{k(2kn+n)-3} \right]$$

$$= \frac{1}{k} \ln(2k+1).$$

于定理23'''令  $k = 4$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{36n-35} + \frac{1}{36n-31} - \frac{8}{36n-27} + \frac{1}{36n-23} \right. \\ \left. + \cdots + \frac{1}{36n-3} \right)$$

$$= \frac{1}{2} \ln 3.$$

于定理23令  $a = k - 6$ ,  $m = k + 1$ , 则有

**定理23<sup>(4)</sup>** 设  $k$  为正整数, 且  $k \geq 7$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k)-6} + \frac{1}{k(kn+n-k+1)-6} \right. \\ \left. + \frac{1}{k(kn+n-k+2)-6} + \cdots + \frac{1}{k(kn+n-7)-6} \right]$$

$$\begin{aligned}
& - \frac{k}{k(kn+n-6)-6} + \frac{1}{k(kn+n-5)-6} + \cdots \\
& + \frac{1}{k(kn+n)-6} \Big] \\
& = \frac{1}{k} \ln(k+1).
\end{aligned}$$

于定理23<sup>(4)</sup>令  $k=7$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left( \frac{1}{56n-55} - \frac{7}{56n-48} + \frac{1}{56n-41} + \cdots + \frac{1}{56n-6} \right) \\
& = \frac{3}{7} \ln 2.
\end{aligned}$$

类似定理17——23这样的结果, 可以给出任意多个.

**定理24** 设  $m$  为正整数, 且  $m \geq 2$ , 则

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} - \frac{m-1}{k(mn-m+1)+a} \right. \\
& \quad + \frac{1}{k(mn-m+2)+a} + \frac{1}{k(mn-m+3)+a} \\
& \quad \left. + \cdots + \frac{1}{k(mn-1)+a} \right] \\
& = C(a, k) - mC(a+k, mk) + \frac{1}{k} \ln \frac{a+k}{a}.
\end{aligned}$$

于定理24令  $a=1$ ,  $m=k+1$ , 则有

**定理24'** 设  $k$  为正整数, 则

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k-1)+1} - \frac{1}{k(kn+n-k)+1} \right. \\
& \quad + \frac{1}{k(kn+n-k+1)+1} + \frac{1}{k(kn+n-k+2)+1} \\
& \quad \left. + \cdots + \frac{1}{k(kn+n-1)+1} \right]
\end{aligned}$$

$$= \frac{1}{k} \ln(k+1).$$

于定理24' 令  $k=1, 2, 3$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{2n} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \ln 2,$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-3)(6n-1)}$$

$$= \frac{1}{8} \sum_{n=1}^{\infty} \left( \frac{1}{6n-5} - \frac{2}{6n-3} + \frac{1}{6n-1} \right)$$

$$= \frac{1}{16} \ln 3,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{3}{12n-8} + \frac{1}{12n-5} + \frac{1}{12n-2} \right)$$

$$= \frac{2}{3} \ln 2.$$

于定理24令  $m=4$ ,  $a=1$ ,  $k=1$ . 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{4n-3} - \frac{3}{4n-2} + \frac{1}{4n-1} + \frac{1}{4n} \right) = 0,$$

令  $m=4$ ,  $a=1$ ,  $k=2$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{8n-7} - \frac{3}{8n-5} + \frac{1}{8n-3} + \frac{1}{8n-1} \right)$$

$$= -\frac{\pi}{2\sqrt{2}} + \frac{\pi}{4} + \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1);$$

令  $m=4$ ,  $a=2$ ,  $k=3$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-10} - \frac{3}{12n-7} + \frac{1}{12n-4} + \frac{1}{12n-1} \right)$$

$$= \frac{\pi}{3\sqrt{3}} - \frac{\pi}{3} - \frac{1}{3} \ln 2 + \frac{1}{\sqrt{3}} \ln(2+\sqrt{3});$$

令  $m=8$ ,  $a=1$ ,  $k=1$ , 即可推得

$$\sum_{n=1}^{\infty} \left( -\frac{1}{8n-7} + \frac{7}{8n-6} - \frac{1}{8n-5} - \cdots - \frac{1}{8n} \right) = \frac{\pi}{2}.$$

**定理25** 设  $m$  为正整数, 且  $m \geq 3$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ & \quad - \frac{m-1}{k(mn-m+2)+a} + \frac{1}{k(mn-m+3)+a} \\ & \quad \left. + \frac{1}{k(mn-m+4)+a} + \cdots + \frac{1}{k(mn-1)+a} \right] \\ & = C(a, k) - mC(a+2k, mk) + \frac{1}{k} \ln \frac{a+2k}{a}. \end{aligned}$$

于定理25令  $a=1$ ,  $m=2k+1$ , 则有

**定理25'** 设  $k$  为正整数, 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(2kn+n-2k-1)+1} + \frac{1}{k(2kn+n-2k)+1} \right. \\ & \quad - \frac{2k}{k(2kn+n-2k+1)+1} + \frac{1}{k(2kn+n-2k+2)+1} \\ & \quad \left. + \cdots + \frac{1}{k(2kn+n-1)+1} \right] \\ & = \frac{1}{k} \ln(2k+1). \end{aligned}$$

于定理25'令  $k=1, 2, 3$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{2}{3n} \right) \\ & = \ln 3, \end{aligned}$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{10n-9} + \frac{1}{10n-7} - \frac{4}{10n-5} + \frac{1}{10n-3} + \frac{1}{10n-1} \right)$$

$$= \frac{1}{2} \ln 5,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{21n-20} + \frac{1}{21n-17} - \frac{6}{21n-14} + \frac{1}{21n-11} \right. \\ \left. + \cdots + \frac{1}{21n-2} \right)$$

$$= \frac{1}{3} \ln 7.$$

于定理25令  $a=2$ ,  $m=k+1$ , 则有

**定理25''** 设  $k$  为正整数, 且  $k \geq 2$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k-1)+2} + \frac{1}{k(kn+n-k)+2} \right. \\ \left. - \frac{k}{k(kn+n-k+1)+2} + \frac{1}{k(kn+n-k+2)+2} \right. \\ \left. + \cdots + \frac{1}{k(kn+n-1)+2} \right] \\ = \frac{1}{k} \ln(k+1).$$

于定理25''令  $k=2, 3, 4$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{2}{3n} \right) \\ = \ln 3,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{12n-10} + \frac{1}{12n-7} - \frac{3}{12n-4} + \frac{1}{12n-1} \right) \\ = \frac{2}{3} \ln 2,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{10n-9} + \frac{1}{10n-7} - \frac{4}{10n-5} + \frac{1}{10n-3} + \frac{1}{10n-1} \right)$$



$$= \frac{1}{2} \ln 5.$$

**定理26** 设  $m$  为正整数, 且  $m \geq 4$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ & \quad + \frac{1}{k(mn-m+2)+a} - \frac{m-1}{k(mn-m+3)+a} \\ & \quad \left. + \frac{1}{k(mn-m+4)+a} + \cdots + \frac{1}{k(mn-1)+a} \right] \\ & = C(a, k) - mC(a+3k, mk) + \frac{1}{k} \ln \frac{a+3k}{a}. \end{aligned}$$

于定理26令  $a=1$ ,  $m=3k+1$ , 则有

**定理26'** 设  $k$  为正整数, 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(3kn+n-3k-1)+1} + \frac{1}{k(3kn+n-3k)+1} \right. \\ & \quad + \frac{1}{k(3kn+n-3k+1)+1} - \frac{3k}{k(3kn+n-3k+2)+1} \\ & \quad \left. + \frac{1}{k(3kn+n-3k+3)+1} + \cdots + \frac{1}{k(3kn+n-1)+1} \right] \\ & = \frac{1}{k} \ln(3k+1). \end{aligned}$$

于定理26'令  $k=1, 2$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{4n-3} + \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{3}{4n} \right) \\ & = \ln 4, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{14n-13} + \frac{1}{14n-11} + \frac{1}{14n-9} - \frac{6}{14n-7} + \frac{1}{14n-5} \right. \\ & \quad \left. + \frac{1}{14n-3} + \frac{1}{14n-1} \right) \end{aligned}$$

$$= \frac{1}{2} \ln 7.$$

于定理26令  $a=3$ ,  $m=k+1$ , 则有

**定理26''** 设  $k$  为正整数, 且  $k \geq 3$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k-1)+3} + \frac{1}{k(kn+n-k)+3} \right. \\ & \quad + \frac{1}{k(kn+n-k+1)+3} - \frac{k}{k(kn+n-k+2)+3} \\ & \quad \left. + \frac{1}{k(kn+n-k+3)+3} + \cdots + \frac{1}{k(kn+n-1)+3} \right] \\ & = \frac{1}{k} \ln(k+1). \end{aligned}$$

于定理26''令  $k=3, 4, 5$ , 则有

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{4n-3} + \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{3}{4n} \right) \\ & = \ln 4, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{20n-17} + \frac{1}{20n-13} + \frac{1}{20n-9} - \frac{4}{20n-5} + \frac{1}{20n-1} \right) \\ & = \frac{1}{4} \ln 5, \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{30n-27} + \frac{1}{30n-22} + \frac{1}{30n-17} - \frac{5}{30n-12} \right. \\ & \quad \left. + \frac{1}{30n-7} + \frac{1}{30n-2} \right) \\ & = \frac{1}{5} \ln 6. \end{aligned}$$

于定理26令  $m=5$ ,  $a=1$ ,  $k=3$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{15n-14} + \frac{1}{15n-11} + \frac{1}{15n-8} - \frac{4}{15n-5} + \frac{1}{15n-2} \right)$$

$$= \frac{\pi}{3\sqrt{3}} + \frac{1}{3} \ln 5,$$

令  $m=5$ ,  $a=1$ ,  $k=8$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{40n-39} + \frac{1}{40n-31} + \frac{1}{40n-23} - \frac{4}{40n-15} + \frac{1}{40n-7} \right) \\ &= \frac{\pi}{4\sqrt{2}} + \frac{1}{8} \ln 5 + \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

令  $m=7$ ,  $a=11$ ,  $k=8$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{56n-45} + \frac{1}{56n-37} + \frac{1}{56n-29} - \frac{6}{56n-21} + \frac{1}{56n-13} \right. \\ & \quad \left. + \frac{1}{56n-5} + \frac{1}{56n+3} \right) \\ &= \frac{\pi}{4\sqrt{2}} - \frac{\pi}{8} + \frac{1}{8} \ln 7 - \frac{1}{3}. \end{aligned}$$

**定理27** 设  $m$  为正整数, 且  $m \geq 5$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} \right. \\ & \quad + \frac{1}{k(mn-m+2)+a} + \frac{1}{k(mn-m+3)+a} \\ & \quad - \frac{m-1}{k(mn-m+4)+a} + \frac{1}{k(mn-m+5)+a} \\ & \quad \left. + \dots + \frac{1}{k(mn-1)+a} \right] \\ &= C(a, m) - mC(a+4k, mk) + \frac{1}{k} \ln \frac{a+4k}{a}. \end{aligned}$$

于定理27令  $a=1$ ,  $m=4k+1$ , 则有

**定理27'** 设  $k$  为正整数, 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(4kn+n-4k-1)+1} + \frac{1}{k(4kn+n-4k)+1} \right]$$

$$\begin{aligned}
& + \frac{1}{k(4kn+n-4k+1)+1} + \frac{1}{k(4kn+n-4k+2)+1} \\
& - \frac{4k}{k(4kn+n-4k+3)+1} + \frac{1}{k(4kn+n-4k+4)+1} \\
& + \cdots + \frac{1}{k(4kn+n-1)+1} \Big] \\
& = \frac{1}{k} \ln(4k+1).
\end{aligned}$$

于定理27' 令  $k=1, 2$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left( \frac{1}{5n-4} + \frac{1}{5n-3} + \frac{1}{5n-2} + \frac{1}{5n-1} - \frac{4}{5n} \right) \\
& = \ln 5,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left( \frac{1}{18n-17} + \frac{1}{18n-15} + \frac{1}{18n-13} + \frac{1}{18n-11} - \frac{8}{18n-9} \right. \\
& \quad \left. + \frac{1}{18n-7} + \cdots + \frac{1}{18n-1} \right) \\
& = \ln 3.
\end{aligned}$$

于定理27令  $a=2$ ,  $m=2k+1$ , 则有

**定理27''** 设  $k$  为正整数, 且  $k \geq 2$ , 则

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{1}{k(2kn+n-2k-1)+2} + \frac{1}{k(2kn+n-2k)+2} \right. \\
& \quad + \frac{1}{k(2kn+n-2k+1)+2} + \frac{1}{k(2kn+n-2k+2)+2} \\
& \quad - \frac{2k}{k(2kn+n-2k+3)+2} + \frac{1}{k(2kn+n-2k+4)+2} \\
& \quad \left. + \cdots + \frac{1}{k(2kn+n-1)+2} \right] \\
& = \frac{1}{k} \ln(2k+1).
\end{aligned}$$

于定理27''令  $k=2, 3$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{5n-4} + \frac{1}{5n-3} + \frac{1}{5n-2} + \frac{1}{5n-1} - \frac{4}{5n} \right) = \ln 5,$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{21n-19} + \frac{1}{21n-16} + \frac{1}{21n-13} + \frac{1}{21n-10} - \frac{6}{21n-7} \right. \\ & \quad \left. + \frac{1}{21n-4} + \frac{1}{21n-1} \right) \\ &= \frac{1}{3} \ln 7. \end{aligned}$$

于定理27令  $a=5$ ,  $m=k+1$ , 则有

**定理27'''** 设  $k$  为正整数, 且  $k \geq 4$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k-1)+4} + \frac{1}{k(kn+n-k)+4} \right. \\ & \quad + \frac{1}{k(kn+n-k+1)+4} + \frac{1}{k(kn+n-k+2)+4} \\ & \quad - \frac{k}{k(kn+n-k+3)+4} + \frac{1}{k(kn+n-k+4)+4} \\ & \quad \left. + \dots + \frac{1}{k(kn+n-1)+4} \right] \\ &= \frac{1}{k} \ln(k+1). \end{aligned}$$

于定理27''', 令  $k=4, 5$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{5n-4} + \frac{1}{5n-3} + \frac{1}{5n-2} + \frac{1}{5n-1} - \frac{4}{5n} \right) = \ln 5,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{36n-26} + \frac{1}{36n-21} + \frac{1}{36n-16} + \frac{1}{36n-11} - \frac{5}{30n-6} \right)$$

$$+ \frac{1}{30n-1}) = \frac{1}{5} \ln 6.$$

于定理27令  $m=5$ ,  $a=3$ ,  $k=8$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{40n-37} + \frac{1}{40n-29} + \frac{1}{40n-21} + \frac{1}{40n-13} - \frac{4}{40n-5} \right) \\ &= \frac{\pi}{4\sqrt{2}} + \frac{1}{8} \ln 5 - \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1); \end{aligned}$$

令  $m=7$ ,  $a=3$ ,  $k=8$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{56n-53} + \frac{1}{56n-45} + \frac{1}{56n-37} + \frac{1}{56n-29} - \frac{6}{56n-21} \right. \\ & \quad \left. + \frac{1}{56n-13} + \frac{1}{56n-5} \right) \\ &= \frac{\pi}{4\sqrt{2}} - \frac{\pi}{8} + \frac{1}{8} \ln 7. \end{aligned}$$

**定理28** 设  $m$  为正整数, 且  $m \geq 6$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} + \dots \right. \\ & \quad + \frac{1}{k(mn-m+4)+a} - \frac{m-1}{k(mn-m+5)+a} \\ & \quad \left. + \frac{1}{k(mn-m+6)+a} + \dots + \frac{1}{k(mn-1)+a} \right] \\ &= C(a, k) - mC(a+5k, mk) + \frac{1}{k} \ln \frac{a+5k}{a}. \end{aligned}$$

于定理28令  $a=1$ ,  $m=5k+1$ , 则有

**定理28'** 设  $k$  为正整数, 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(5kn+n-5k-1)+1} + \frac{1}{k(5kn+n-5k)+1} + \dots \right. \\ & \quad \left. + \frac{1}{k(5kn+n-5k+3)+1} - \frac{5k}{k(5kn+n-5k+4)+1} \right] \end{aligned}$$

$$+ \frac{1}{k(5kn+n-5k+5)+1} + \cdots + \frac{1}{k(5kn+n-1)+1} \Big] \\ = \frac{1}{k} \ln(6k+1).$$

于定理28' 令  $k=1$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{6n-5} + \frac{1}{6n-4} + \frac{1}{6n-3} + \frac{1}{6n-2} + \frac{1}{6n-1} - \frac{5}{6n} \right) \\ = \ln 6.$$

于定理28令  $a=5$ ,  $m=k+1$ , 则有

**定理28''** 设  $k$  为正整数, 且  $k \geq 5$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k-1)+5} + \frac{1}{k(kn+n-k)+5} + \cdots \right. \\ \left. + \frac{1}{k(kn+n-k+3)+5} - \frac{k}{k(kn+n-k+4)+5} \right. \\ \left. + \frac{1}{k(kn+n-k+5)+5} + \cdots + \frac{1}{k(kn+n-1)+5} \right] \\ = \frac{1}{k} \ln(k+1).$$

于定理28'' 令  $k=5, 6$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{6n-5} + \frac{1}{6n-4} + \frac{1}{6n-3} + \frac{1}{6n-2} + \frac{1}{6n-1} - \frac{5}{6n} \right) \\ = \ln 6,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{42n-37} + \frac{1}{42n-31} + \cdots + \frac{1}{42n-13} - \frac{6}{42n-7} \right. \\ \left. + \frac{1}{42n-1} \right)$$

$$= \frac{1}{6} \ln 7.$$

于定理28令  $m=7$ ,  $a=1$ ,  $k=4$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{28n-27} + \frac{1}{28n-23} + \cdots + \frac{1}{28n-11} - \frac{6}{28n-7} \right. \\ & \quad \left. + \frac{1}{28n-3} \right) \\ &= \frac{\pi}{4} + \frac{1}{4} \ln 7. \end{aligned}$$

**定理29** 设  $m$  为正整数, 且  $m \geq 7$ , 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(mn-m)+a} + \frac{1}{k(mn-m+1)+a} + \cdots \right. \\ & \quad + \frac{1}{k(mn-m+5)+a} - \frac{m-1}{k(mn-m+6)+a} \\ & \quad \left. + \frac{1}{k(mn-m+7)+a} + \cdots + \frac{1}{k(mn-1)+a} \right] \\ &= C(a, k) - mC(a+6k, mk) + \frac{1}{k} \ln \frac{a+6k}{a}. \end{aligned}$$

于定理29令  $a=1$ ,  $m=6k+1$ , 则有

**定理29'** 设  $k$  为正整数, 则

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ \frac{1}{k(6kn+n-6k-1)+1} + \frac{1}{k(6kn+n-6k)+1} + \cdots \right. \\ & \quad + \frac{1}{k(6kn+n-6k+4)+1} - \frac{6k}{k(6kn+n-6k+5)+1} \\ & \quad \left. + \frac{1}{k(6kn+n-6k+6)+1} + \cdots + \frac{1}{k(6kn+n-1)+1} \right] \\ &= \frac{1}{k} \ln(6k+1). \end{aligned}$$

于定理29'令  $k=1, 2$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{7n-6} + \frac{1}{7n-5} + \cdots + \frac{1}{7n-1} - \frac{6}{7n} \right)$$



$$= \ln 7,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{26n-25} + \frac{1}{26n-23} + \cdots + \frac{1}{26n-15} - \frac{12}{26n-13} \right. \\ \left. + \frac{1}{26n-11} + \cdots + \frac{1}{26n-1} \right) \\ = \frac{1}{2} \ln 13.$$

于定理29令  $a=2$ ,  $m=3k+1$ , 则有

**定理29''** 设  $k$  为正整数, 且  $k \geq 2$ , 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(3kn+n-3k-1)+2} + \frac{1}{k(3kn+n-3k)+2} + \cdots \right. \\ \left. + \frac{1}{k(3kn+n-3k+4)+2} - \frac{3k}{k(3kn+n-3k+5)+2} \right. \\ \left. + \frac{1}{k(3kn+n-3k+6)+2} + \cdots + \frac{1}{k(3kn+n-1)+2} \right] \\ = \frac{1}{k} \ln(3k+1).$$

于定理29''令  $k=2, 3$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{7n-6} + \frac{1}{7n-5} + \cdots + \frac{1}{7n-1} - \frac{6}{7n} \right) \\ = \ln 7,$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{30n-28} + \frac{1}{30n-25} + \cdots + \frac{1}{30n-13} - \frac{9}{30n-10} \right. \\ \left. + \frac{1}{30n-7} + \frac{1}{30n-4} + \frac{1}{30n-1} \right) \\ = \frac{1}{3} \ln 10.$$

于定理29令  $a=3$ ,  $m=2k+1$ , 则有

**定理29'''** 设  $k$  为正整数, 且  $k \geq 3$ , 则

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{1}{k(2kn+n-2k-1)+3} + \frac{1}{k(2kn+n-2k)+3} + \dots \right. \\
& \quad + \frac{1}{k(2kn+n-2k+4)+3} - \frac{2k}{k(2kn+n-2k+5)+3} \\
& \quad \left. + \frac{1}{k(2kn+n-2k+6)+3} + \dots + \frac{1}{k(2kn+n-1)+3} \right] \\
& = \frac{1}{k} \ln(2k+1).
\end{aligned}$$

于定理29'''令  $k=4$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left( \frac{1}{36n-33} + \frac{1}{36n-29} + \dots + \frac{1}{36n-13} - \frac{8}{36n-9} \right. \\
& \quad \left. + \frac{1}{36n-5} + \frac{1}{36n-1} \right) \\
& = \frac{1}{2} \ln 3.
\end{aligned}$$

于定理29令  $a=6$ ,  $m=k+1$ , 则有

**定理29<sup>(4)</sup>** 设  $k$  为正整数, 且  $k \geq 6$ , 则

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{1}{k(kn+n-k-1)+6} + \frac{1}{k(kn+n-k)+6} + \dots \right. \\
& \quad + \frac{1}{k(kn+n-k+4)+6} - \frac{k}{k(kn+n-k+5)+6} \\
& \quad \left. + \frac{1}{k(kn+n-k+6)+6} + \dots + \frac{1}{k(kn+n-1)+6} \right] \\
& = \frac{1}{k} \ln(k+1).
\end{aligned}$$

于定理29<sup>(4)</sup>令  $k=7$ , 则有

$$\sum_{n=1}^{\infty} \left( \frac{1}{56n-50} + \frac{1}{56n-43} + \dots + \frac{1}{56n-15} - \frac{7}{56n-8} \right)$$

$$+ \frac{1}{56n-1})$$

$$= \frac{3}{7} \ln 2.$$

于定理29令  $m=7$ ,  $a=1$ ,  $k=8$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{56n-55} + \frac{1}{56n-47} + \cdots + \frac{1}{56n-15} - \frac{6}{56n-7} \right)$$

$$= \frac{\pi}{4\sqrt{2}} + \frac{\pi}{8} + \frac{1}{8} \ln 7.$$

类似定理24——29这样的结果, 可以给出任意多个.

**定理30** 设  $m$  为正整数, 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{k(2mn+n-2m-1)+a} + \frac{1}{k(2mn+n-2m)+a} \right.$$

$$+ \frac{1}{k(2mn+n-2m+1)+a} + \cdots$$

$$+ \frac{1}{k(2mn+n-m-2)+a} - \frac{2m}{k(2mn+n-m-1)+a}$$

$$+ \frac{1}{k(2mn+n-m)+a} + \frac{1}{k(2mn+n-m+1)+a}$$

$$\left. + \cdots + \frac{1}{k(2mn+n-1)+a} \right]$$

$$= C(a, k) - (2m+1)C(a+mk, 2mk+k)$$

$$+ \frac{1}{k} \ln \frac{a+mk}{a}.$$

于定理30令  $a=1$ ,  $k=2$ , 则有

**定理30'** 设  $m$  为正整数, 则

$$\sum_{n=1}^{\infty} \left[ \frac{1}{(4m+2)n-4m-1} + \frac{1}{(4m+2)n-4m+1} \right]$$

$$\begin{aligned}
& + \frac{1}{(4m+2)n-4m+3} + \cdots + \frac{1}{(4m+2)n-2m-3} \\
& - \frac{2m}{(4m+2)n-2m-1} + \frac{1}{(4m+2)n-2m+1} \\
& + \frac{1}{(4m+2)n-2m+3} + \cdots + \frac{1}{(4m+2)n-1} \Big] \\
& = \frac{1}{2} \ln(2m+1).
\end{aligned}$$

于定理30' 令  $m=1, 2$ , 则有

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(6n-5)(6n-3)(6n-1)} \\
& = \frac{1}{8} \sum_{n=1}^{\infty} \left( \frac{1}{6n-5} - \frac{2}{6n-3} + \frac{1}{6n-1} \right) \\
& = \frac{1}{16} \ln 3,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left( \frac{1}{10n-9} + \frac{1}{10n-7} - \frac{4}{10n-5} + \frac{1}{10n-3} + \frac{1}{10n-1} \right) \\
& = \frac{1}{2} \ln 5.
\end{aligned}$$

于定理30 令  $m=1, a=1, k=1$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)3n} = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{3n-2} - \frac{2}{3n-1} + \frac{1}{3n} \right) \\
& = \frac{\pi}{4\sqrt{3}} - \frac{1}{4} \ln 3,
\end{aligned}$$

令  $m=1, a=1, k=4$ , 即可推得

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{(12n-11)(12n-7)(12n-3)} \\
& = \frac{1}{32} \sum_{n=1}^{\infty} \left( \frac{1}{12n-11} - \frac{2}{12n-7} + \frac{1}{12n-3} \right)
\end{aligned}$$

$$= \frac{\pi\sqrt{3}}{256} - \frac{\pi}{256} - \frac{1}{256} \ln 3 + \frac{\sqrt{3}}{128} \ln(2 + \sqrt{3}),$$

令  $m=1$ ,  $a=1$ ,  $k=8$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(24n-23)(24n-15)(24n-7)} \\ &= \frac{1}{128} \sum_{n=1}^{\infty} \left( \frac{1}{24n-23} - \frac{2}{24n-15} + \frac{1}{24n-7} \right) \\ &= \frac{\pi}{1024} + \frac{1}{1024} \ln 3 + \frac{1}{256\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

令  $m=1$ ,  $a=7$ ,  $k=8$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{(24n-17)(24n-9)(24n-1)} \\ &= \frac{1}{128} \sum_{n=1}^{\infty} \left( \frac{1}{24n-17} - \frac{2}{24n-9} + \frac{1}{24n-1} \right) \\ &= -\frac{\pi}{1024} + \frac{1}{1024} \ln 3 + \frac{1}{256\sqrt{2}} \ln(\sqrt{2}+1), \end{aligned}$$

令  $m=2$ ,  $a=4$ ,  $k=3$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{15n-11} + \frac{1}{15n-8} - \frac{4}{15n-5} + \frac{1}{15n-2} + \frac{1}{15n+1} \right) \\ &= \frac{\pi}{3\sqrt{3}} + \frac{1}{3} \ln 5 - 1; \end{aligned}$$

令  $m=2$ ,  $a=7$ ,  $k=4$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{20n-13} + \frac{1}{20n-9} - \frac{4}{20n-5} + \frac{1}{20n-1} + \frac{1}{20n+3} \right) \\ &= \frac{1}{4} \ln 5 - \frac{1}{3}, \end{aligned}$$

令  $m=2$ ,  $a=9$ ,  $k=8$ , 即可推得

$$\sum_{n=1}^{\infty} \left( \frac{1}{40n-31} + \frac{1}{40n-23} - \frac{4}{40n-15} + \frac{1}{40n-7} + \frac{1}{40n+1} \right)$$

$$= \frac{\pi}{4\sqrt{2}} + \frac{1}{8} \ln 5 + \frac{1}{2\sqrt{2}} \ln(\sqrt{2} + 1) - 1,$$

令  $m=2$ ,  $a=1$ ,  $k=12$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{60n-59} + \frac{1}{60n-47} - \frac{4}{60n-35} + \frac{1}{60n-23} \right. \\ & \quad \left. + \frac{1}{60n-11} \right) \\ &= \frac{\pi}{4\sqrt{3}} + \frac{1}{12} \ln 5 + \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}); \end{aligned}$$

令  $m=2$ ,  $a=11$ ,  $k=12$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{60n-49} + \frac{1}{60n-37} - \frac{4}{60n-25} + \frac{1}{60n-13} + \frac{1}{60n-1} \right) \\ &= -\frac{\pi}{4\sqrt{3}} + \frac{1}{12} \ln 5 + \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}); \end{aligned}$$

令  $m=3$ ,  $a=11$ ,  $k=8$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{56n-45} + \frac{1}{56n-37} + \frac{1}{56n-29} - \frac{6}{56n-21} \right. \\ & \quad \left. + \frac{1}{56n-13} + \frac{1}{56n-5} + \frac{1}{56n+3} \right) \\ &= \frac{\pi}{4\sqrt{2}} - \frac{\pi}{8} + \frac{1}{8} \ln 7 - \frac{1}{8}; \end{aligned}$$

令  $m=3$ ,  $a=13$ ,  $k=12$ , 即可推得

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \frac{1}{84n-71} + \frac{1}{84n-59} + \frac{1}{84n-47} - \frac{6}{84n-35} + \frac{1}{84n-23} \right. \\ & \quad \left. + \frac{1}{84n-11} + \frac{1}{84n+1} \right) \\ &= \frac{\pi}{6} + \frac{1}{12} \ln 7 + \frac{1}{2\sqrt{3}} \ln(2 + \sqrt{3}) - 1. \end{aligned}$$

[ G e n e r a l   I n f o r m a t i o n ]

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